On Tree Amplitudes in Gauge Theory and Gravity

Jared Kaplan (with Nima Arkani-Hamed)

Outline

- I Motivation
- II Review of BCFW Recursion Relations
  - A Analytic Continuation of Momenta
  - B Physical Interpretation as a Very Hard Particle in a Soft Background
- III Understanding BCFW Limit for YM and Gravity via...
  - A Scalar QED and Yang-Mills
  - B Yang-Mills
  - C A Photon with Gravity
  - D Gravity

Motivation

Amplitudes are simpler than Feynman Diagrams suggest.

- Gauge/Gravity amplitudes contain few terms, despite $n!$ diagrams $\implies$ extensive cancelations
- Existence of BCFW Recursion Relations for amplitudes

Motivation

Amplitudes are simpler than Feynman Diagrams suggest.

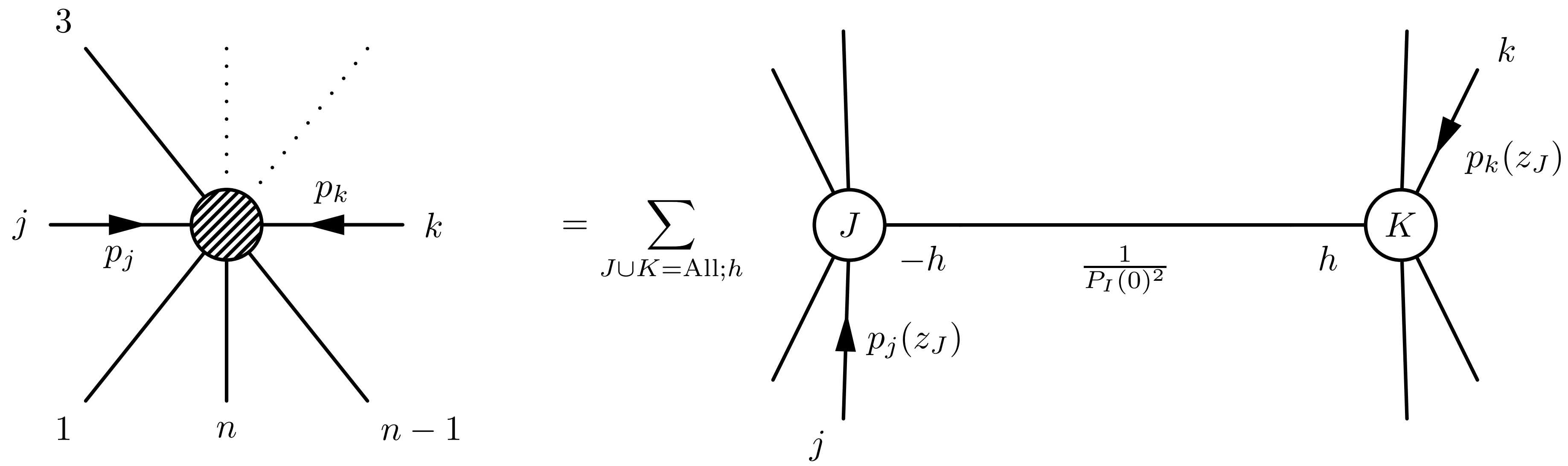
- Gauge/Gravity amplitudes contain few terms, despite $n!$ diagrams $\implies$ extensive cancelations
- Existence of BCFW Recursion Relations for amplitudes

And there are deeper reasons to care...

- With Gravity, local observables are imprecise
- Diagrams contain off-shell info – un-physical and local
- Holography in flat space = S-Matrix Theory?

What Are BCFW Recursion Relations?

Amplitudes are written as sums of products of lower amplitudes with appropriate propagators.



Review of BCFW Recursion Relations

Very Schematically:

$$\mathcal{M}_n = \sum_r \mathcal{M}_{r+1} \frac{1}{P_r^2} \mathcal{M}_{n-r+1}$$

The BCFW Recursion Relations are

- true for gauge theory and gravity
- generalizable to many other renormalizable QFTs
- independent of twistors and the spinor-helicity formalism
- a property of QFT in any number of dimensions $D \geq 4$

These points are new.

Review of BCFW Recursion Relations

The idea of the derivation:

- tree amplitudes are simply rational functions of momenta
- rational functions are determined by their poles
- We must find a ‘nice’ analytic continuation that exposes all physical poles, without spurious poles

Review of BCFW Recursion Relations

Consider an amplitude $M(p_i, h_i)$. Pick legs j and k and analytically continue on-shell:

$$p_j \rightarrow p_j(z) = p_j + qz \quad \text{and} \quad p_k \rightarrow p_k(z) = p_k - qz$$

with

$$q \cdot p_{j,k} = 0, \quad q^2 = 0$$

Specifically, we can choose

$$p_j = (1, 1, 0, 0; 0.., 0),\ p_k = (1, -1, 0, 0; 0, ..0),\ q = (0, 0, 1, i; 0, ..0)$$

Review of BCFW Recursion Relations

In Gauge Theory or Gravity, polarization tensors?

In a helicity basis, naturally have

$$\epsilon_j^- = \epsilon_k^+ = q, \quad \epsilon_j^+ = \epsilon_k^- = q^*, \quad \epsilon_T = (0,0,0,0,...,1,...,0)$$

Must analytically continue to

$$\begin{aligned} \epsilon_j^-(z) &= \epsilon_k^+(z) = q \\ \epsilon_j^+(z) &= q^* - z p_k \\ \epsilon_k^-(z) &= q^* + z p_j \\ \epsilon_T(z) &= (0,0,0,0,...,1,...,0) \end{aligned}$$

Review of BCFW Recursion Relations

Now we have $\mathcal{M}(z)$. At tree, $\mathcal{M}(z)$ *only has simple poles*

$$\frac{1}{P_J(z)^2} = \frac{1}{\left(\sum_{i\in J} p_i\right)^2} = \frac{1}{P_J(0)^2 - 2zq\cdot P_J}$$

Residues at poles are a product of lower amplitudes!

$$\begin{aligned}\mathrm{res}\mathcal{M}(z\to z_J) &= \sum_h \mathcal{M}(i\in J, p_i(z_J), h_i; -P_J(z_J), h) \\ &\quad \times\mathcal{M}(i\notin J, p_i(z_J), h_i; P_J(z_J), -h)\end{aligned}$$

Review of BCFW Recursion Relations

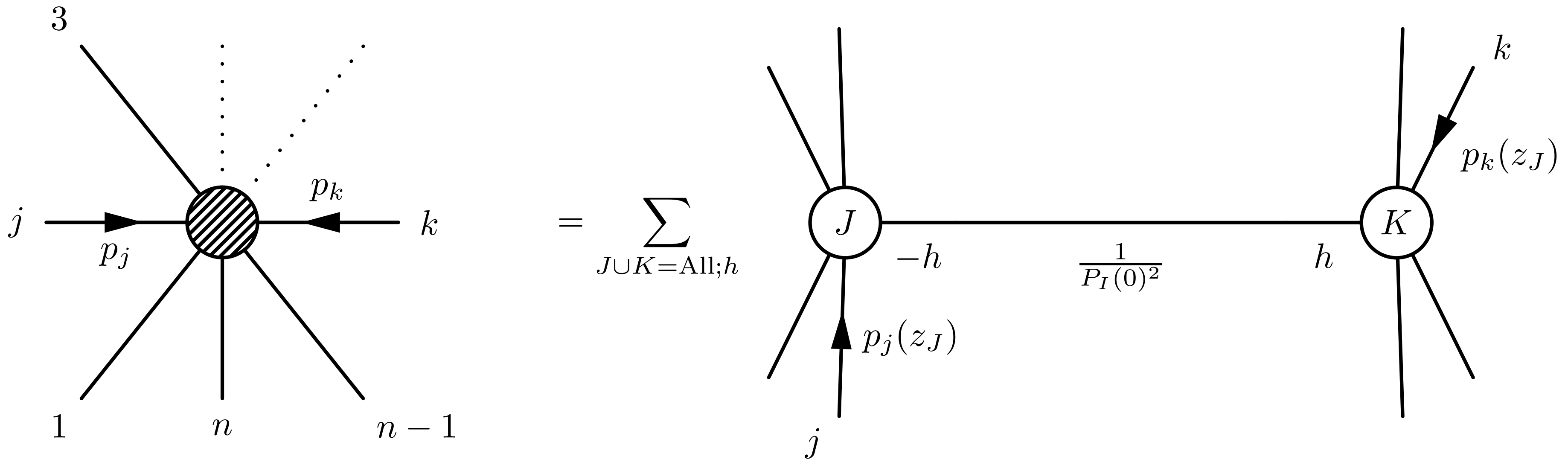
If $\mathcal{M}(z) \to 0$ as $z \to \infty$ – a very non-trivial property – then

$$0 = \int dz/z M(z) = M(0) + \text{residues}$$

giving the BCFW Recursion Relations:

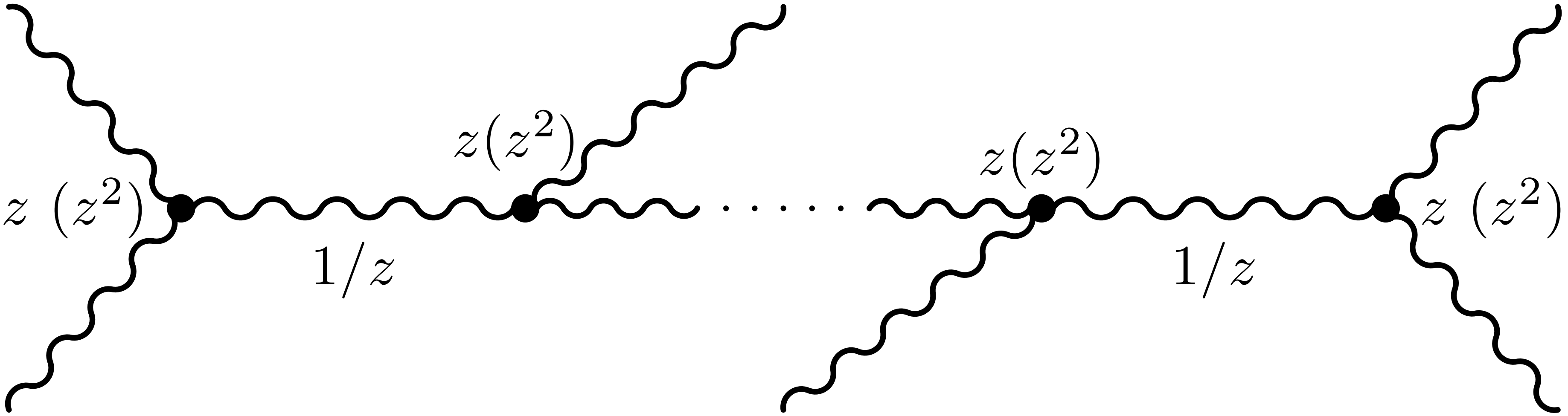
$$\begin{aligned} M(0) &= \sum_{J,h} M(i \in J, p_i(z_J), h_i; -P_J(z_J), h) \\ &\times \frac{1}{P_J^2} \times M(i \notin J, p_i(z_J), h_i; P_J(z_J), -h) \end{aligned}$$

The BCFW Recursion Relations



Surprising Behavior of $\mathcal{M}(z \to \infty)$

Does $\mathcal{M}(z) \to 0$ as $z \to \infty$? Naively, never!



Also, for ϕ^4, find that $\mathcal{M}(z) \to$ constant.

Surprising Behavior of $\mathcal{M}(z \to \infty)$

In what follows, we will show that for many helicity combinations, $\mathcal{M}(z) \to 0$ as $z \to \infty$. In gauge theory

$$M^{-+}(z),\ M^{--}(z),\ M^{++}(z) \to \frac{1}{z} \quad \text{and} \quad M^{+-} \to z^3$$

Also, gravity z-dependence $=$ gauge theory squared.

Note that we only need ($-$, any) to vanish for Recursion.

Meaning of $\mathcal{M}(z \to \infty)$

In the limit $z \to \infty$, we have $p_j(z), p_k(z) \to \infty$.

This is a very hard massless particle in a soft background.

Thus we need only study the quadratic lagrangian of a particle with background gauge/gravity turned on.

We will study Scalar QED, Scalar Yang-Mills, Yang-Mills, Scalar Gravity, Photon-Gravity, and finally Gravity...

Outline of the Analysis

(1) For scalar QED and Yang-Mills, we use an obvious gauge choice, but we will run into a rather non-trivial subtlety.

(2) For Yang-Mills, we will need to understand the symmetry structure of the BCFW limit. We will also use the Ward Identity.

(3) For gravity, we need all of these ingredients, plus a KLT-inspired field re-definition.

$\mathcal{M}(z \to \infty)$ for Scalar QED

Amplitude with 2 scalars, n photons; continue scalar momenta.

$$L = D_\mu \phi^* D^\mu \phi$$

Naively, $M(z) \to z$ since there is an $O(p^\mu)$ vertex.

Want to consider large momentum – but momentum isn't gauge invariant!?

Can strip off a Wilson line...

$\mathcal{M}(z \to \infty)$ for Scalar QED

Write

$$\phi(x) \quad = \quad W_n(x)\tilde{\phi}(x)$$

where

$$W_n(x) = \exp\left(i \int_{-\infty}^{0} d\lambda \; n_\mu A^\mu(x + \lambda n) \right)$$

Then the lagrangian becomes

$$L = (W_n \partial_\mu \tilde{\phi} + D_\mu W_n \tilde{\phi})^* (W_n \partial^\mu \tilde{\phi} + D^\mu W_n \tilde{\phi})$$

$\mathcal{M}(z \to \infty)$ for Scalar QED

Now the only terms that can grow as $z \to \infty$ are

$$\partial_\mu \tilde{\phi}^* \tilde{\phi} W_n^* D^\mu W_n \to i z q_\mu \tilde{\phi}^* \tilde{\phi} W_n^* D^\mu W_n$$

But the gauge invariant combination

$$q_\mu W_n^* D^\mu W_n = i \int_{-\infty}^{0} d\lambda \, q_\mu F^{\mu\nu}(x + n\lambda) n_\nu$$

If we choose $n^\mu = q^\mu$, then this vanishes due to the anti-symmetry of $F^{\mu\nu}$.

Thus we have shown in a gauge-invariant way that there are no physical $O(z)$ vertices.

$\mathcal{M}(z \to \infty)$ for Scalar QED

Of course, there is an easier way – choose a gauge!

Since only $O(z)$ interactions come from $zq \cdot A$, natural gauge

$$q \cdot A = A_- = 0$$

q-light cone gauge. Only have scalar propagators $\mathcal{O}(1/z)$, no z-dependent vertices, so

$$\mathcal{M}_2(z) \to z^0 \quad \text{and} \quad M_{n>2}(z) \to \frac{1}{z}$$

Have Recursion Relations for $n > 2$ photons!

$\mathcal{M}(z \to \infty)$ for Scalar Yang-Mills

Unfortunately, this argument was too quick. Although in

$$q_\mu W_n^* D^\mu W_n = i \int_{-\infty}^{0} d\lambda\, q_\mu F^{\mu\nu}(x + n\lambda) n_\nu$$

the integrand vanishes as $n \to q$, the integral is over an infinite line, so we might have $0 \times \infty$.

In momentum space

$$\begin{aligned} \left[q_\mu W_n^* D^\mu W_n\right](p) &= \frac{q_\mu F^{\mu\nu}(p) n_\nu}{p \cdot n} \\ &= \frac{(A(p) \cdot q)(p \cdot n) - (A(p) \cdot n)(p \cdot q)}{p \cdot n} \end{aligned}$$

$\mathcal{M}(z \to \infty)$ for Scalar Yang-Mills

If $p \cdot q = 0$, then we find $0/0$ as $n \to q$, so

$$\lim_{n \to q} \left[q_\mu W_n^* D^\mu W_n \right](p) = A(p) \cdot q$$

and there is an $O(z)$ vertex.

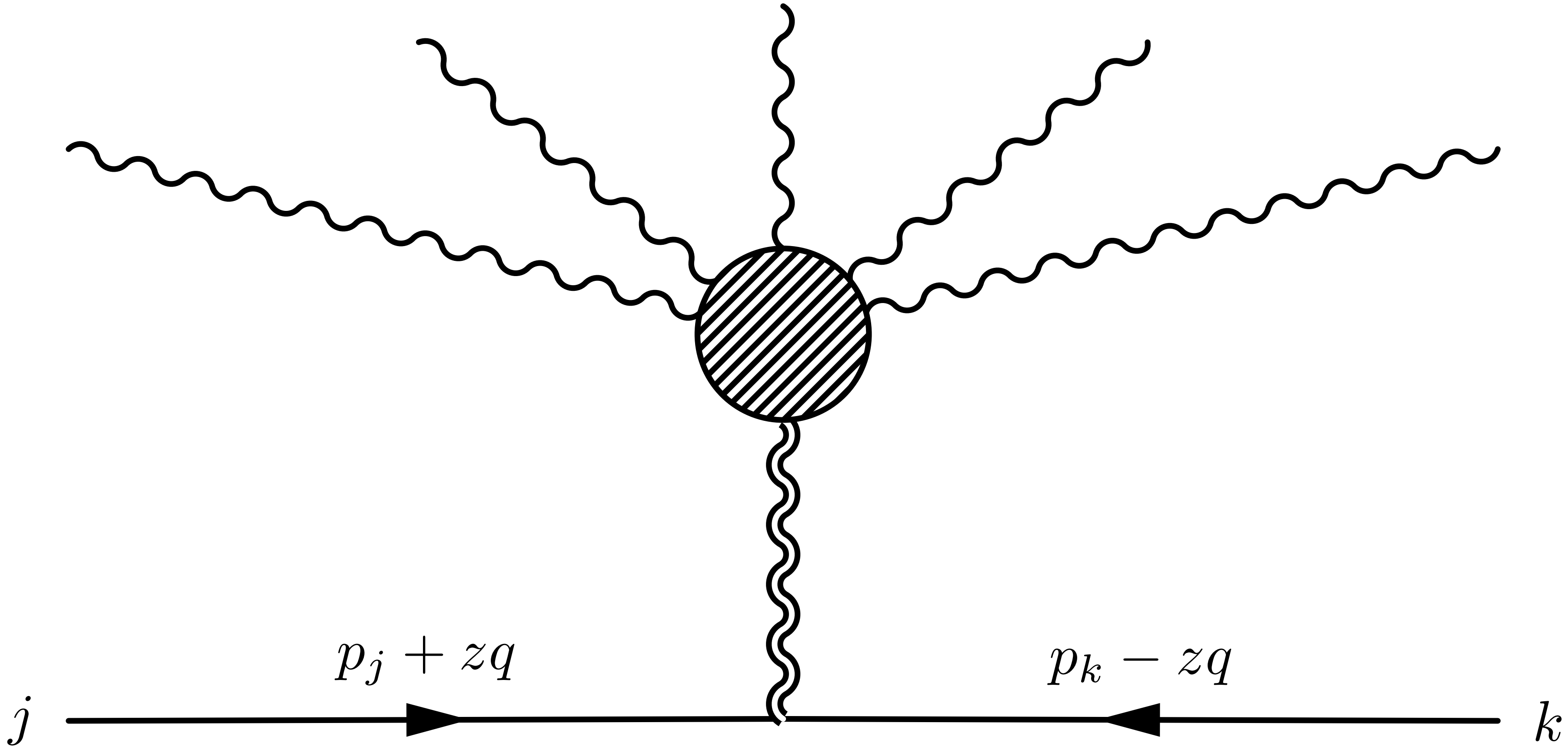
This shows up as an inability to choose light cone gauge:

$$q^\mu A_\mu(p) + i q^\mu p_\mu \Lambda(p) = 0$$

If $p \cdot q$ vanishes, we cannot change gauge.

With a background $A(p)$ with $p \cdot q = 0$, have $O(z)$ vertex.

A Dangerous Diagram



Only dangerous if the background is self-interacting – i.e. Yang-Mills or Gravity. No problem for scalar QED.

Yang-Mills Theory

The full amplitude can be written as

$$\mathcal{M} = \epsilon_j^a \mathcal{M}_{ab} \epsilon_k^b$$

We will compute an ansatz for $\mathcal{M}_{ab}$ at large z.

Along with the Ward identity

$$p_{ja}(z) \mathcal{M}^{ab} \epsilon_{kb} = 0$$

this will be enough to prove BCFW.

Yang-Mills Theory

The lagrangian for a fluctuation about a background is

$$L = -\frac{1}{4}\,\mathrm{tr} D_{[\mu}a_{\nu]}D^{[\mu}a^{\nu]} + \frac{i}{2}\mathrm{tr}\,[a_\mu, a_\nu]F^{\mu\nu}$$

where D is A-covariant. We have two gauge freedoms.

Fix a gauge freedom with usual gauge fixing term:

$$L = -\frac{1}{4}\mathrm{tr}\,\eta^{ab}D_\mu a_a D^\mu a_b + \frac{i}{2}\mathrm{tr}[a_a, a_b]F^{ab}$$

Use η^{ab} to emphasize *spin-lorentz symmetry*.

Yang-Mills Theory

$$L = -\frac{1}{4}\mathrm{tr}\,\eta^{ab} D_\mu a_a D^\mu a_b + \frac{i}{2}\mathrm{tr}[a_a, a_b] F^{ab}$$

Only first term can give $O(z)$ vertex, but $\propto \eta^{ab}$.

Get $O(1)$ from single insertion of anti-symmetric F^{ab}.

Further insertions give $O(1/z)$ or smaller. Thus

$$\mathcal{M}^{ab} = (cz + \cdots)\eta^{ab} + A^{ab} + \frac{1}{z}B^{ab} + \cdots$$

with A^{ab} anti-symmetric.

Computing Using the Ward Identity

Now we compute $\mathcal{M}(z \to \infty)$ with the Ward Identity:

$$p_{ja}(z) M^{ab} \epsilon_{kb} = 0$$

Thus since $p_{ja}(z) = p_j + zq$, we have

$$q_a M^{ab} \epsilon_{kb} = -\frac{1}{z} p_{ja} M^{ab} \epsilon_{kb}$$

This is extremely useful, because $\epsilon_j^- = \epsilon_k^+ = q$!

An Example Computation

$$\begin{aligned}
M^{--}(z) &= \epsilon^{-}_{ja} M^{ab} \epsilon^{-}_{kb} \\
&= -\frac{1}{z} p_{ja} \left[(cz + \cdots)\eta^{ab} + A^{ab} + \frac{1}{z} B^{ab} \right] (q^{*}_{b} + z p_{jb}) \\
&= -\frac{1}{z} p_{ja} A^{ab} q^{*}_{b} - \frac{1}{z} p_{ja} B^{ab} p_{jb} + O(1/z^{2}) \\
&\to \frac{1}{z}
\end{aligned}$$

Previously, this result was derived using MHV diagrams, which are special to 4-d.

Gauge Theory Results

$\epsilon_1 \backslash \epsilon_2$	$-$	$+$	T
$-$	$1/z$	$1/z$	$1/z$
$+$	z^3	$1/z$	z
T1	z	$1/z$	z
T2	z	$1/z$	1

We see that $(-$,any$)$ all vanish, so the BCFW Recursion Relations obtain.

Scalar Gravity, Briefly

$$L = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

We can choose light-cone gauge using diff invariance:

$$g^{++} = g^{+i} = 0 \quad \text{and} \quad g^{+-} = 1$$

This eliminates all $O(z^2)$ vertices except the unique, ‘subtle’ and ‘dangerous’ diagram, so $\mathcal{M}(z) \to z^2$.

Photon-Graviton Amplitudes

Introduce vielbein

$$\begin{aligned} g_{\mu\nu} &= e_{\mu a} e_{\nu b} \eta^{ab} \\ \omega_{\mu ab} &= e_a^{\nu} \nabla_\mu e_{\nu b}. \end{aligned}$$

Very useful because we want $\mathcal{M}^{ab}$ Lorentz.

With vielbein, have

$$\begin{aligned} A_\mu &= e_\mu^a A_a \\ \nabla_\nu A_\mu &= e_\mu^a D_\nu A_a \\ D_\nu A_a &= \partial_\nu A_a + \omega_{\nu a}{}^c A_c \end{aligned}$$

Photon-Graviton Amplitudes

Thus the Gauge Fixed Photon Lagrangian is

$$L = -\sqrt{-g} g^{\mu\nu} \eta^{ab} (\partial_\mu A_a + \omega_{\mu a}{}^{c} A_c)(\partial_\nu A_b + \omega_{\nu b}{}^{d} A_d)$$

Spin Lorentz, broken by anti-symmetric $\omega_{\mu ac}$.

Choose q-light cone gauge for background:

$$g^{++} = g^{+i} = 0,\ g^{+-} = 1 \ \text{ and } \omega^{+}_{ab} = 0$$

Photon-Graviton Amplitudes

$$L = -\sqrt{-g}g^{\mu\nu}\eta^{ab}(\partial_\mu A_a + \omega_{\mu a}{}^c A_c)(\partial_\nu A_b + \omega_{\nu b}{}^d A_d)$$

With gauge choice, only $O(z^2)$ and $O(z)$ vertices from ‘dangerous’ diagram, so

$$M^{ab} = cz^2\eta^{ab} + zA^{ab} + B^{ab} + \cdots$$

with A^{ab} anti-symmetric.

Could compute amplitudes, but let’s do gravity...

Gravity Amplitudes

De-Donder gauge quadratic Lagrangian with Dilaton

$$\begin{aligned} L &= \sqrt{-g}\left[\frac{1}{4}g^{\mu\nu}\nabla_\mu h^\beta_\alpha \nabla_\nu h^\alpha_\beta - \frac{1}{8}g^{\mu\nu}\nabla_\mu h^\alpha_\alpha \nabla_\nu h^\beta_\beta \right. \\ &\left. -h_{\alpha\beta}h_{\mu\nu}\frac{1}{2}R^{\beta\mu\alpha\nu} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \end{aligned}$$

Trace terms are dangerous for us; seem to violate KLT

$$\mathrm{Gravity} = \mathrm{Gauge} \times \mathrm{Gauge}$$

Gravity Amplitudes

Use field re-defition to eliminate traces:

$$h_{\mu\nu} \to h_{\mu\nu} + g_{\mu\nu}\sqrt{\frac{2}{D-2}}\phi, \qquad \phi \to \frac{1}{2}g^{\mu\nu}h_{\mu\nu} + \sqrt{\frac{D-2}{2}}\phi$$

Thus we have

$$L = \sqrt{-g}\left[\frac{1}{4}g^{\mu\nu}g^{\alpha\rho}g^{\beta\sigma}\nabla_\mu h_{\alpha\beta}\nabla_\nu h_{\rho\sigma} - \frac{1}{2}h_{\alpha\beta}h_{\mu\nu}R^{\beta\mu\alpha\nu}\right]$$

where we have dropped the dilaton.

Gravity Amplitudes

Now introduce vielbein to make $\mathcal{M}^{a\tilde{a}b\tilde{b}}$ Lorentz:

$$h_{\mu\nu} = e^a_\mu \tilde{e}^{\tilde{a}}_\nu h_{a\tilde{a}}, \; \nabla_\alpha h_{\mu\nu} = e^a_\mu \tilde{e}^{\tilde{a}}_\nu D_\alpha h_{a\tilde{a}} \qquad (1)$$

with

$$D_\alpha h_{a\tilde{a}} = \partial_\alpha h_{a\tilde{a}} + \omega^{b}_{\alpha a} h_{b\tilde{a}} + \tilde{\omega}^{\tilde{b}}_{\alpha\tilde{a}} h_{a\tilde{b}}$$

We have introduced a ‘left’ and ‘right’ vielbein to emphasize that these indices are never coupled, but of course

$$\tilde{e} = e, \quad \tilde{\omega} = \omega$$

Gravity Amplitudes

Lagrangian becomes

$$L = \sqrt{-g}\left[\frac{1}{4} g^{\mu\nu}\eta^{ab}\tilde{\eta}^{\tilde{a}\tilde{b}} D_\mu h_{a\tilde{a}} D_\nu h_{b\tilde{b}} - \frac{1}{2} h_{a\tilde{a}} h_{b\tilde{b}} R^{ab\tilde{a}\tilde{b}}\right]$$

with

$$D_\alpha h_{a\tilde{a}} = \partial_\alpha h_{a\tilde{a}} + \omega^{b}_{\alpha a} h_{b\tilde{a}} + \tilde{\omega}^{\tilde{b}}_{\alpha\tilde{a}} h_{a\tilde{b}}$$

Choose q-light cone gauge

$$\omega^{+}_{ab} = \tilde{\omega}^{+}_{\tilde{a}\tilde{b}} = g^{++} = g^{+i} = 0 \quad \text{and} \quad g^{+-} = 1$$

Gravity Amplitudes

Thus we find the ansatz

$$\begin{aligned} M^{a\tilde{a}b\tilde{b}} &= cz^2\eta^{ab}\tilde{\eta}^{\tilde{a}\tilde{b}} + z\left(\eta^{ab}\tilde{A}^{\tilde{a}\tilde{b}} + A^{ab}\tilde{\eta}^{\tilde{a}\tilde{b}}\right) \\ &\quad + A^{ab\tilde{a}\tilde{b}} + \eta^{ab}\tilde{B}^{\tilde{a}\tilde{b}} + B^{ab}\tilde{\eta}^{\tilde{a}\tilde{b}} + \frac{1}{z}C^{ab\tilde{a}\tilde{b}} + \cdots \end{aligned}$$

where the A terms are anti-symmetric.

This is precisely the ‘square’ of the gauge theory ansatz.

It is sufficient to prove BCFW for any $D \geq 4$.

Conclusions and Future Directions

- We have shown BCFW for Gravity and Gauge Theory with $D \geq 4$, deriving z-dependence for all helicities.
- Continue more momenta – Get ϕ^4 theory...
- Loop level Recursion Relations?
- Relation to $N = 4$ SYM, $N = 8$ Sugra – amplitudes determined by unitarity cuts
- Find direct S-Matrix Derivation, Re-formulate QFT?