Massive Gravity

Kurt Hinterbichler

February 26, 2008

Contents

1 Introduction

These are notes on massive gravity intended to fill in the details of my journal club talk, for those who may be interested. Fields in flat four dimensional space are categorized by the spins of the particles they carry. If we wish to describe long range forces, only bosonic fields will do, since fermionic fields cannot build up classical coherent states. By the spin statistics theorem, these must be integer spin fields $s = 0, 1, 2, \ldots$. A field of mass m will satisfy the Klein-Gordon equation $(\partial^2 - m^2)\psi_i = 0$, whose solution around a source will look like $\sim \frac{1}{r}e^{-mr}$, so long range forces should be described by massless fields, $m = 0$.

Massless particles are characterized by how they transform under the little group $ISO(2)$ (the isometries of the plane), that part of the Poincare group leaving a reference lightlike four-momentum invariant. The representation is required to be unitary and finite dimensional, so it must represent the non-compact part of $ISO(2)$ trivially, leaving just an $SO(2)$ representation characterized by an integer ≥ 0, which we call the helicity. For helicity 0, such massless particles can be carried most simply by a scalar field ϕ. There is no gauge symmetry, and any sort of interaction terms cubic and higher in the field can be consistently added.

For helicities $s \geq 1$, the actions must carry a gauge symmetry. This is because the fields used to carry such higher helicities contain non trivial representations of the non-compact part of $ISO(2)$, which the gauge symmetry mods out. For helicity 1, if we choose a vector field A_μ to carry the particle, its action is fixed to be the Maxwell action, if no other degrees of freedom are desired. The form of the gauge symmetry is also fixed. If Maxwell had known that the photon was a spin one particle, he could have arrived at EM much more straightforwardly via group theory. If we now ask for consistent self interactions of such massless particles, we are led to the problem of deforming the action (and possibly the form of the gauge transformations), in such a way that preserves the number of gauge transformations and their form at linear level. This is an interesting problem in BRST cohomology, and leads us essentially uniquely to the non-abelian gauge theories [18].

For helicity 2, the action and gauge symmetries are also fixed. We will look at this action in section 2. If we ask for consistent self interactions, we are led essentially uniquely to GR. For helicity ≥ 3, there are essentially no self interactions that can be written.

These theories of massless particles are very nice. We will see in section 3 how great GR is. It is a model effective field theory with a cutoff at the Planck mass, M_p. Around heavy sources, there is a classical linear regime, where r is greater than the the Schwartzschild radius, $r > \frac{M}{M_p^2} \sim r_S$. For M the mass of the Sun, we have $r_S \sim 1$ km, so the linear approximation is good everywhere in the solar system. Then there is a well separated classical non-linear regime, $\frac{1}{M_p} < r < r_S$, where non-linearities can be summed up without worrying about quantum corrections. This regime can be used to make controlled statements about what is going on inside a black hole. Quantum effect do not become important until $r < \frac{1}{M_p}$, very near the singularity of the black hole.

Of course, even if we accept these massless theories, it is natural to wonder whether the particles involved could actually have a small mass. A massive spin zero particle has one degree of freedom, the same as a massless helicity zero particle. There is no problem whatsoever adding a small mass term to a massless scalar field action, and all physical quantities are continuous in the mass. Indeed this is what we would like physically. Surely we should not be able to say that a parameter in nature, like a particle mass, is exactly mathematically zero. We should only place small limits on what the mass could be, as would be the case if measurable quantities are continuous in the mass as $m \to 0$.

However, things are not so simple with the higher spins. Adding a small mass for the photon changes the theory to that of a massive spin one particle, which has three degrees of freedom. Thus the massless limit is inherently not continuous. We will see in section 6.1 that a smooth massless limit can be defined, and that these three degrees of freedom become the two degrees of freedom of a massless helicity 1, and the one degree of freedom of a massless helicity 0. It can also be seen that the helicity 0 decouples, so if we take this as the massless limit, we still have continuity in physical predictions.

If we try to add a mass in the helicity 2 case, the situation is even worse. A massive spin 2 particle has five degrees of freedom. Defining a smooth limit as before, we find that these become the two degrees of freedom of a helicity 2, the two degrees of freedom of a helicity 1, and the one degree of freedom of a helicity 0. The helicity 1 decouples, but the helicity 0 does not, and a discontinuity in the predictions of the linear theory remains. This is the vDVZ discontinuity, after van Dam, Veltman and Zakharov, which we'll study in detail from several points of view.

If the linear theory is accurate, then the vDVZ discontinuity represents a true physical discontinuity in predictions. Massive gravity in the $m \to 0$ limit gives a prediction for light bending that is off by 25 percent from the GR prediction, and measuring this would be a way to show that the graviton mass is mathematically zero rather than just very small. However, it was noticed by Vainstein that the linear approximation for a graviton of mass m actually breaks down at a huge distance from a source, called the Vainstein radius $r_V = \left(\frac{GM}{m^4}\right)^{1/5}$, where M is the mass of the source. This radius goes to infinity as $m \to 0$, so there is no radius at which the linear approximation tells us something trustworthy about the massless limit. This opens the possibility that non-linear effects cure the discontinuity. If we take M the mass of the sun, and m a very small value, say the Hubble constant $m \sim 10^{-33}$eV, the scale at which we might want to modify gravity to explain the consmological constant, we have $r_V \sim 10^{18}$km, about the size of the Milky Way (A light year is $\sim 10^{13}$km).

It is still unclear whether non-linear effects do in fact cure the discontinuity in the case of classical massive gravity. There are other models, such as DGP, which modify gravity in a way similar to adding a mass term. There is a discontinuity analogous to the vDVZ discontinuity at linear level, and it can be seen explicitly thought exact non-perturbative solutions that it is cured at non-linear level.

However, quantum mechanically the situation is much worse. Adding a small mass to GR can be thought of as a very mindless infrared modification of the theory. It brutally violates the elegant gauge symmetries and throws in new degrees of freedom. The price to pay is that the cutoff is lowered from M_p down to the scale $\Lambda_5 = (M_p m^4)^{1/5}$. For Hubble scale graviton mass, this is $\Lambda_5^{-1} \sim 10^{11}$m. As such, the quantum effects become important at the radius $r_* = \left(\frac{M}{M_{Pl}}\right)^{1/3} \frac{1}{\Lambda_5}$, which is parametrically larger than the Vainshtein radius at which non-linearities enter. For this sun we have $r_* \sim 10^{21}$km. Without finding a UV completion, there is no sense in which we can trust the solution inside this radius, and no hope to examine the continuity of physical quantities in m.

The situation can be improved somewhat by introducing an infinite number of higher order interactions in such a way that the cutoff is raised to $\Lambda_3 = (M_p m^2)^{1/3}$, $\Lambda_3^{-1} \sim 10^5$m. However, a ghost with mass below the cutoff appears around heavy source solutions even in the classical regioin, so either the cutoff must be lowered again, or the background is unstable (this ghost also appears in the original theory with cutoff Λ_5, but its mass is above the cutoff in the classical region). These ghosts can be traced to a sixth degree of freedom, which is present in the full non-lineary massive gravity theory, but absent at linear level.

Thus, the conclusion is that studying massive gravity is essentially worthless, in the sense that one cannot extract reliable predictions from it (of course, this doesn't imply that it is not correct). Infrared modifications of gravity are still interesting, because of the possibility that the cosmological constant has some dynamical origin. However, if any predictions are to be made, we should probably be looking at more clever modifications that preserve the symmetries of GR, such as DGP.

An outline of these notes is as follows. In section 2 we study the action for a massless helicity two graviton, and solve its equations of motion around a point source. In section 3, we look at GR from the point of view of effective field theory, and admire how nice it is. In section 4, we study the action for a massive spin two graviton, solve its equations of motion around a point source, and compare to the massless case to exhibit the vDVZ discontinuity. In section 5, we deform GR by adding a mass term, then solve explicitly the non-linear equations around a point source to second order in non-linearity, to show directly that the linear solution breaks down at the Vainshtein radius. In section 6, we see the vDVZ discontinuity at linear level from another point of view, by introducing the Stükelberg trick, useful for explicitly displaying the degrees of freedom responsible for the discontinuity. In section 7, we extend the Stükelberg trick to full non-linear level, using an elegant geometric picture, and use it in section 8 to study the effective theory of massive GR. We will see that both the non-linearity and the small cutoff of the effective theory are due to strong coupling of the longitudinal degree of freedom of the graviton.

2 Massless helicity 2

We use the mostly plus metric convention. A spin 2 particle in D-dimensional flat space is carried by a symmetric tensor field $h_{\mu\nu}$. The action is

$$S_{\text{linear}} = \int d^D x - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h + \kappa h_{\mu\nu}T^{\mu\nu}. \quad (2.1)$$

where we've added a symmetric source $T^{\mu\nu}$ for $h_{\mu\nu}$,The normalization $+\frac{1}{2}h_{\mu\nu}T^{\mu\nu}$ is in accord with the general relativity definition $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}$, as well as the normalization $\delta g_{\mu\nu} = 2\kappa h_{\mu\nu}$.

The action is determined by group theory. The coefficients are tuned so that the equations of motion will be a projection operator onto the helicity 2 part of the field. The equations of motion are

$$\frac{\delta S_{\text{linear}}}{\delta h^{\mu\nu}} =$$

$$\boxed{\partial^2 h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^2 h + \kappa T_{\mu\nu} = 0.} \quad (2.2)$$

By acting on the equations of motion with ∂_μ, we find that the left side vanishes identically, and so the source must be conserved if there are to be any solutions to the equations of motion,

$$\partial_\mu T^{\mu\nu} = 0. \quad (2.3)$$

The action, like the action for any massless particle of spin ≥ 1, is gauge invariant. It is invariant under the gauge transformations

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (2.4)$$

(The reasons for these gauge symmetries are also group theoretical. See the discussion in the introduction, and Weinberg volume 1.)

We often choose the Lorentz gauge (also called harmonic, or deDonder gauge),

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \quad (2.5)$$

This condition fixes the gauge only up to gauge transformations with parameter ξ_μ satisfying $\partial^2 \xi_\mu = 0$. In this gauge, the equations of motion simplify to

$$\partial^2 h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial^2 h = -\kappa T_{\mu\nu}. \quad (2.6)$$

Taking the trace, we find, assuming $D \neq 2$,

$$\partial^2 h = \frac{2}{D-2}\kappa T, \quad (2.7)$$

and upon substituting back, we get

$$\partial^2 h_{\mu\nu} = -\kappa \left[T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}T\right]. \quad (2.8)$$

This equation, along with the Lorentz gauge condition, is equivalent to the original equation of motion in Lorentz gauge.

2.1 Propagator in Lorentz gauge

Taking ∂^μ on 2.8 and on its trace, using conservation of $T_{\mu\nu}$ and comparing, we have $\partial^2(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h) = 0$, so that the lorentz condition is automatically satisfied when appropriate boundary conditions are satisfied so that $\partial^2 f = 0 \Rightarrow f = 0$ for any function f. We can then solve 2.8 by fourier transforming.

$$T^{\mu\nu}(p) = \int d^D x \; e^{ipx} \; T^{\mu\nu}(x), \tag{2.9}$$

$$h_{\mu\nu}(x) = \kappa \int \frac{d^D p}{(2\pi)^D} \; e^{ipx} \frac{1}{p^2} \left[T_{\mu\nu}(p) - \frac{1}{D-2} \eta_{\mu\nu} T(p) \right]. \tag{2.10}$$

2.2 Field of a point source

Now, specialize to four dimensions, and consider as source the stress tensor of a mass M particle at rest at the origin

$$T^{\mu\nu}(x) = M \delta^\mu_0 \delta^\nu_0 \delta^3(\mathbf{x}), \quad T^{\mu\nu}(p) = 2\pi M \delta^\mu_0 \delta^\nu_0 \delta(p^0). \tag{2.11}$$

The general solution reduces to

$$h_{\mu\nu}(x) = \kappa \int \frac{d^4 p}{(2\pi)^4} e^{i\mathbf{px}} \frac{1}{\mathbf{p}^2} \left[\delta^0_\mu \delta^0_\nu - \frac{1}{2} \eta_{\mu\nu}(-1) \right] (2\pi M) \delta(p^0). \tag{2.12}$$

so that

$$\begin{aligned} h_{00}(x) &= \frac{\kappa M}{2} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} e^{i\mathbf{px}} \frac{1}{\mathbf{p}^2} = \frac{\kappa M}{2} \frac{1}{4\pi r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{\kappa M}{2} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} e^{i\mathbf{px}} \frac{1}{\mathbf{p}^2} \delta_{ij} = \frac{\kappa M}{2} \frac{1}{4\pi r} \delta_{ij}. \end{aligned} \tag{2.13}$$

Keeping in mind the newtonian relation $2\kappa h_{00} = -2\phi$, $2\kappa h_{ij} = -2\psi \delta_{ij}$, and $\kappa^2 = 8\pi G$ we have for the newtonian potential and spatial components

$$\begin{aligned} \phi &= -\frac{GM}{r}, \\ \psi &= -\frac{GM}{r}. \end{aligned} \tag{2.14}$$

The PPN parameter is $\gamma = 1$ and the magnitude of the light bending angle for light incident at impact parameter b is

$$\alpha = \frac{4GM}{b}. \tag{2.15}$$

3 GR

If we now ask for the massless spin 2 particle to have self interactions, we must add higher order term in such a way that the gauge invariance is preserved (the form of the gauge transformations may be altered to allow higher terms in the non-linearities). Such an extension is essentially unique, and leads to the action for general relativity,

$$S_{\text{GR}} = \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-g}R. \tag{3.1}$$

The action is invariant under general diffeomorphisms $f^\mu(x)$,

$$g_{\mu\nu} \to \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu}g_{\alpha\beta}\left(f(x)\right).$$

Infinitesimally, for $f^\mu(x) = x^\mu + \xi^\mu(x)$, we find the gauge transformations

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu. \tag{3.2}$$

where ξ^μ is the gauge parameter and indices are lowered by the metric.

The field equation for the metric is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

To see that this is an extension of the massless spin 2 field, we expand the action around the flat space solution $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

The second variation is

$$S_2 = \frac{1}{2\kappa^2}\int d^Dx\,\frac{1}{2}\delta^2(\sqrt{-g}R), \tag{3.3}$$

$$\delta^2(\sqrt{-g}R) = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h \tag{3.4}$$

where indices on $h_{\mu\nu}$ are raised and traced with the flat background metric $\eta_{\mu\nu}$ and we have ignored total derivatives. After scaling

$$h_{\mu\nu} \to 2\kappa h_{\mu\nu}$$

the linear action for GR is exactly that of the massless spin two particle in Minkowski space

$$S_{\text{GR linear}} = \int d^Dx - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h. \tag{3.5}$$

It is invariant under the linearized GR gauge transformations

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{3.6}$$

If we continue the expansion around flat space to higher non-linear order in $h_{\mu\nu}$, we have, schematically,

$$\mathcal{L}_{\rm GR} = \partial^2 h^2 + \kappa h \partial^2 h^2 + \cdots + \kappa^n h^n \partial^2 h^2 + \cdots \tag{3.7}$$

All interaction terms have two derivatives, and higher and higher powers of $h_{\mu\nu}$ are suppressed by appropriate powers of κ.

Around any background, the gauge transformations are modified by non-linearities only at first order in $h_{\mu\nu}$,

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \mathcal{L}_\xi h_{\mu\nu} \tag{3.8}$$

This is an all orders expression in $h_{\mu\nu}$.

3.1 Spherical solution, breakdown of linearity

We attempt to find spherically symmetric solutions to the equations of motion

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{3.9}$$

The most general spherically symmetric static metric can be written

$$g_{\mu\nu} = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2. \tag{3.10}$$

The most general gauge transformation which preserves this ansatz is a reparametrization of the radial coordinate r. We can use this to eliminate the function $A(r)$, bringing the metric into the form

$$g_{\mu\nu} = -B(r)dt^2 + C(r)\left[dr^2 + r^2 d\Omega^2\right]. \tag{3.11}$$

The linear expansion of this around flat space will be seen to correspond to the Lorentz gauge choice. Plugging this ansatz into the equations of motion, we get the following from the tt equation and rr equation respectively,

$$3r\left(C'\right)^2 - 4C\left(2C' + rC''\right) = 0 \tag{3.12}$$

$$4B'C^2 + 2\left(2B + rB'\right)C'C + Br\left(C'\right)^2 = 0. \tag{3.13}$$

The $\theta\theta$ equation, (which is the same as the $\phi\phi$ equation by spherical symmetry) turns out to be redundant. It is implied by the tt and rr equations (this happens because of a Noether identity resulting from the radial re-parametrization gauge invariance).

We start by doing a linear expansion of these equations around the flat space solution

$$B_0(r) = 1, \quad C_0(r) = 1. \tag{3.14}$$

We do this by the usual method of linearizing a non-linear differential equation about a solution. We introduce the expansion

$$\begin{aligned} B(r) &= B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots , \\ C(r) &= C_0(r) + \epsilon C_1(r) + \epsilon^2 C_2(r) + \cdots , \end{aligned} \tag{3.15}$$

where ϵ will be a parameter that counts the order of non-linearity. We proceed by plugging into the equations of motion and collecting like powers of ϵ. The $\mathcal{O}(0)$ part gives $0 = 0$ because B_0, C_0, A_0 are solutions to the full non-linear equations. At each higher order in ϵ we will obtain a linear equation that lets us solve for the next term in terms of the solutions to previous terms.

At $\mathcal{O}(\epsilon)$ we obtain

$$C_1'' + \frac{2C_1}{r} = 0, \quad B_1' + C_1' = 0. \tag{3.16}$$

There are three arbitrary constants in the general solution. Demanding that B_1 and C_1 go to zero as $r \to \infty$, so that the solution is asymptotically flat, fixes two. The other constant remains unfixed, and represents the mass of the black hole solution. We choose it to reproduce the solution we got from the propagator. We have then,

$$B_1 = -\frac{2GM}{r}, \quad C_1 = \frac{2GM}{r}. \tag{3.17}$$

At $\mathcal{O}(\epsilon^2)$ we obtain another set of differential equations

$$\frac{3G^2M^2}{r^4} - \frac{2C_2'}{r} - C_2'' = 0 \tag{3.18}$$

$$\frac{7G^2M^2}{r^3} + B_2' + C_2' = 0. \tag{3.19}$$

Again there are three arbitrary constants in the general solution. Demanding that B_2 and C_2 go to zero as $r \to \infty$ again fixes two. The third appears as the coefficient of a $\frac{1}{r}$ term, and we set it to zero so that the second order term doesn't compete with the first order as $r \to \infty$. We can continue in this way to any order, and we obtain the expansion

$$B(r) - 1 = -\frac{2GM}{r}\left(1 - \frac{GM}{r} + \cdots\right), \tag{3.20}$$

$$C(r) - 1 = \frac{2GM}{r}\left(1 + \frac{3GM}{4r} + \cdots\right). \tag{3.21}$$

$$\tag{3.22}$$

The dots represent higher powers in the non-linearity ϵ. We see that the non-linearlity expansion is an expansion in the parameter r_S/r, where

$$r_S = 2GM, \tag{3.23}$$

is the Schwartzschild radius.

In fact, this expansion can be summed to all orders by solving the original equations exactly,

$$B(r) = \frac{\left(1 - \frac{2r}{GM}\right)^2}{\left(1 + \frac{2r}{GM}\right)^2}, \qquad C(r) = (1 + \frac{GM}{2r})^4.$$

This is the Schwartzschild solution, in Lorentz gauge.

3.2 GR as a quantum effective field theory

We can understand the previous results from an effective field theory viewpoint, and check that the black hole solution we obtained is still valid despite quantum corrections. Take the Einstein action expanded around flat space and add a source term.

$$\mathcal{L}_{\mathrm{GR}} = \partial^2 h^2 + \frac{1}{M_p} h \partial^2 h^2 + \cdots + \frac{1}{M_p^n} h^n \partial^2 h^2 + \cdots + \frac{1}{M_p} hT \tag{3.24}$$

Classically, we want to calculate h around a point source of mass M, in which case we are looking at tree graphs.

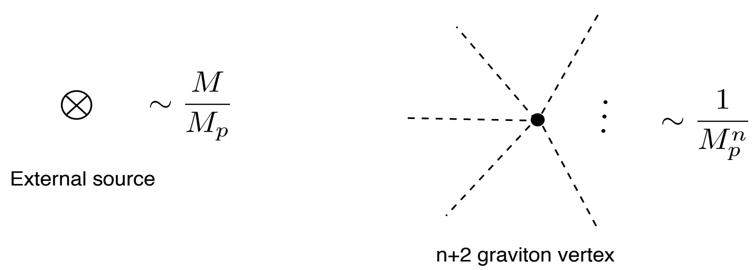

Figure 1: Perturbation theory for finding GR solutions.

Each external source gets one power of $\frac{M}{M_p}$, each n-point vertex gets one power of $\frac{1}{M_p^{n-2}}$, and the power of r is obtained by dimensional analysis. To linear order we have

$$h \sim \frac{M}{M_p} \frac{1}{r}. \tag{3.25}$$

In fact, any graph with n external currents will go like

$$\left(\frac{M}{M_p}\right)^n \frac{1}{M_p^{n-1} r^n}. \tag{3.26}$$

We see that each higher order is suppressed from the order before it by the factor

$$\frac{M}{M_p^2 r} \sim \frac{GM}{r}, \tag{3.27}$$

so that when $r < GM$ the perturbation theory breaks down and non-linear effects become important.

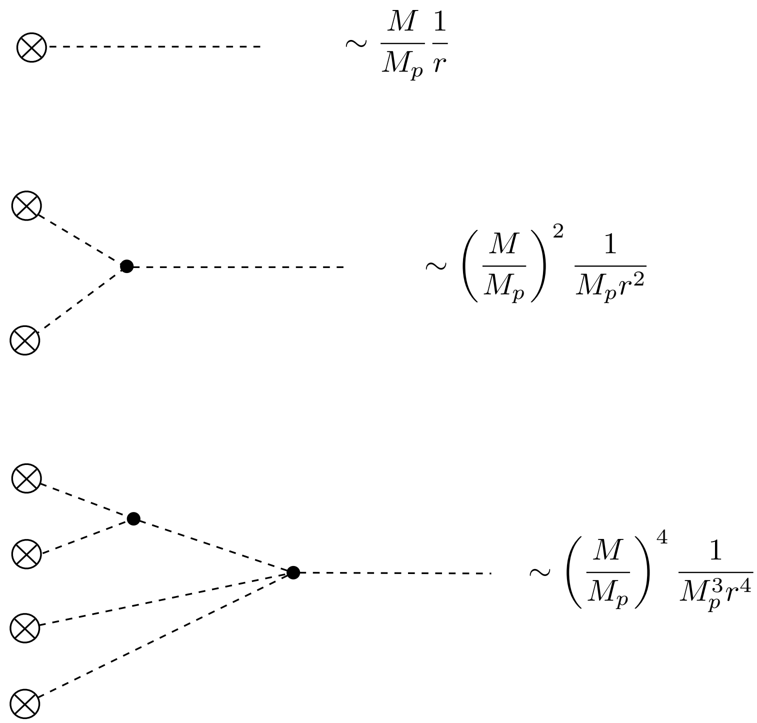

Figure 2: Values for some diagrams.

Quantum mechanically, we also expect to generate a whole slew of other operators in the quantum effective action. By gauge invariance, all operators with two derivatives should sum up to $\sqrt{-g}R$. However we can generate operators with different numbers of derivatives, suppressed by appropriate powers of the plank scale, for example,

$$\frac{1}{M_p^2}\partial^4 h^2, \quad \frac{1}{M_p^3}\partial^4 h^3, \quad \frac{1}{M_p^5}\partial^6 h^3, \quad M_p^2 h^2, \cdots \tag{3.28}$$

By gauge invariance, they must sum up to curvature scalars, times appropriate powers of M_p,

$$M_p^4\sqrt{-g} \sim M_p^4 + M_p^3 h + M_p^2 h^2 + \cdots \tag{3.29}$$

$$\sqrt{-g}R^2 \sim \frac{1}{M_p^2}\partial^4 h^2 + \frac{1}{M_p^3}\partial^4 h^3 + \frac{1}{M_p^4}\partial^4 h^4 + \cdots \tag{3.30}$$

$$\frac{1}{M_p^2}\sqrt{-g}R\nabla^2 R \sim \frac{1}{M_p^4}\partial^6 h^2 + \frac{1}{M_p^5}\partial^6 h^3 + \cdots \tag{3.31}$$

$$\frac{1}{M_p^2}\sqrt{-g}R^3 \sim \frac{1}{M_p^5}\partial^6 h^3 + \frac{1}{M_p^6}\partial^6 h^4 + \cdots \tag{3.32}$$

$$\tag{3.33}$$

These corrections include terms second order in the fields, but higher order in the derivatives. Higher derivative terms such as this always lead to new degrees of freedom, some of which are ghosts or tachyons, and one might worry why these terms are generated here. However, the masses of these ghosts and tachyons is always near or above the cutoff M_p, so they should not be considered part of the effective theory. Any UV completion should cure them. They must not be re-summed into the propagator (this would be stepping outside the M_p expansion), but rather treated as vertices in the effective theory.

Inclusion of any one of these quantum vertices into a tree graph with n external sources will generate a correction to the n graph with vertices drawn only from R. However, this correction will always be down by powers of M_p from the classical graphs. Thus they only become important when

$$r \sim \frac{1}{M_p}. \tag{3.34}$$

Thus there is this huge middle regime, where the theory becomes non-perturbative, and yet quantum effects are still small. We can re-sum the linear expansion by solving the full classical einstein equations, ignoring the quantum corrections, and trust the results down to the plank length. The scale of non-linearity is well separated from the quantum scale.

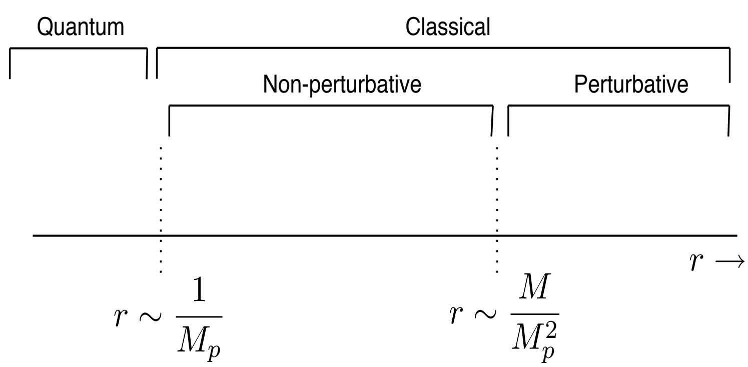

Figure 3: Regimes for GR.

4 Massive spin 2

Suppose we think that the graviton is massive. The action for a single massive spin 2 particle in flat space is again determined by group theory.

$$S_{\text{m linear}} = \int d^Dx - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \tag{4.1}$$

The structure of the mass term takes the Fierz-Pauli form $h_{\mu\nu}h^{\mu\nu}-h^2$. Any deviation from this form and the action will no longer describe a single massive spin two particle– it will have extra pathological degrees of freedom [1, 2]. The massive spin 2 action has no gauge symmetry, the mass term breaks the gauge symmetry possessed by the massless spin 2 action.

4.1 Solution to the linear equation

Add a symmetric source $T^{\mu\nu}$ for $h_{\mu\nu}$,

$$S_{\text{m linear}} = \int d^Dx - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu}-h^2) + \kappa h_{\mu\nu}T^{\mu\nu}. \tag{4.2}$$

Despite the absence of gauge symmetry, we still assume that the source is conserved,

$$\partial_\mu T^{\mu\nu} = 0. \tag{4.3}$$

This assumption is open to challenge, since conservation was a consistency constraint coming from the gauge invariance of the massless action, and there is no gauge invariance here, hence no consistency constraint.

The equations of motion are

$$\frac{\delta S_{\text{m linear}}}{\delta h^{\mu\nu}} =$$

$$\boxed{\partial^2 h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^2 h - m^2(h_{\mu\nu}-\eta_{\mu\nu}h) + \kappa T_{\mu\nu} = 0.} \tag{4.4}$$

Acting on the equations of motion with ∂^μ, we find, assuming $m^2 \neq 0$,

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = 0. \tag{4.5}$$

Plugging this back into the equations of motion, we find

$$\partial^2 h_{\mu\nu} - \partial_\lambda\partial_\nu h^\lambda{}_\mu - m^2(h_{\mu\nu}-\eta_{\mu\nu}h) = -\kappa T_{\mu\nu}.$$

Taking the trace of this, and again applying (4.5), we find

$$m^2(D-1)h = -\kappa T,$$

Assuming $D \neq 1$ we have

$$h = -\frac{\kappa}{m^2(D-1)}T. \tag{4.6}$$

Applying this to (4.5), we find

$$\partial^\mu h_{\mu\nu} = -\frac{\kappa}{m^2(D-1)}\partial_\nu T, \tag{4.7}$$

which when applied along with (4.6) to the equations of motion, implies

$$(\partial^2 - m^2)h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)T\right]. \tag{4.8}$$

Thus we have seen that the equations of motion imply the three equations,

$$\boxed{\begin{gathered}(\partial^2 - m^2)h_{\mu\nu} = -\kappa \left[T_{\mu\nu} - \tfrac{1}{D-1}\left(\eta_{\mu\nu} - \tfrac{\partial_\mu\partial_\nu}{m^2}\right)T\right] \\ \partial^\mu h_{\mu\nu} - \partial_\nu h = 0 \\ h = -\tfrac{\kappa}{m^2(D-1)}T.\end{gathered}} \tag{4.9}$$

Conversely, it is easy to see that these three equations imply the equations of motion, so they are equivalent. The first of these equations is an evolution equation for the $D(D+1)/2$ components of a symmetric tensor, and the last two are constraint equations. The last determines the trace completely, killing one degree of freedom. The second gives D initial value constraints, whose preservation in time implies D more initial value constraints, thus killing D degrees of freedom. In total, we are left with the $(D+1)(D-2)/2$ degrees of freedom of a D-dimensional "spin 2" particle.

4.2 Propagator

Taking the first of 4.9 and tracing, we see that under the assumption that $(\partial^2 - m^2)f = 0 \Rightarrow f = 0$ for any function f, the third equation is implied. This will be the case with good boundary conditions. The second equation can also be shown to follow under this assumption, so that we can obtain the solution by fourier transforming only the first equation.

The general solution for a given a source is,

$$h_{\mu\nu}(x) = \kappa \int \frac{d^D p}{(2\pi)^D}\, e^{ipx} \frac{1}{p^2 + m^2}\left[T_{\mu\nu}(p) - \frac{1}{D-1}\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)T(p)\right], \tag{4.10}$$

where $T^{\mu\nu}(p)$ is the fourier transform of the source

$$T^{\mu\nu}(p) = \int d^D x\, e^{ipx}\, T^{\mu\nu}(x). \tag{4.11}$$

4.3 Field of a point source

Now, specialize to four dimensions, and consider as source the stress tensor of a mass M particle at rest at the origin

$$T^{\mu\nu}(x) = M\delta^\mu_0\delta^\nu_0\delta^3(\mathbf{x}), \quad T^{\mu\nu}(p) = 2\pi M\delta^\mu_0\delta^\nu_0\delta(p^0). \tag{4.12}$$

The general solution reduces to

$$h_{\mu\nu}(x) = \kappa \int \frac{d^4 p}{(2\pi)^4} e^{i\mathbf{px}} \frac{1}{\mathbf{p}^2 + m^2}\left[\delta^0_\mu\delta^0_\nu - \frac{1}{3}\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)(-1)\right](2\pi M)\delta(p^0). \tag{4.13}$$

so that

$$\begin{aligned} h_{00}(x) &= \frac{2\kappa M}{3}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{1}{\mathbf{p}^2+m^2}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{\kappa M}{3}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{1}{\mathbf{p}^2+m^2}\left(\delta_{ij}+\frac{p_ip_j}{m^2}\right). \end{aligned} \tag{4.14}$$

Using the formulae

$$\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{1}{\mathbf{p}^2+m^2} = \frac{1}{4\pi}\frac{e^{-mr}}{r}, \tag{4.15}$$

$$\begin{aligned} \int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{p_ip_j}{\mathbf{p}^2+m^2} &= -\partial_i\partial_j\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{1}{\mathbf{p}^2+m^2} \\ &= \frac{1}{4\pi}\frac{e^{-mr}}{r}\left[\frac{1}{r^2}(1+mr)\delta_{ij}-\frac{1}{r^4}(3+3mr+m^2r^2)x_ix_j\right], \end{aligned} \tag{4.16}$$

we have

$$\begin{aligned} h_{00}(x) &= \frac{2\kappa M}{3}\frac{1}{4\pi}\frac{e^{-mr}}{r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{\kappa M}{3}\frac{1}{4\pi}\frac{e^{-mr}}{r}\left[\frac{1+mr+m^2r^2}{m^2r^2}\delta_{ij}-\frac{1}{m^2r^4}(3+3mr+m^2r^2)x_ix_j\right]. \end{aligned} \tag{4.17}$$

Using the following conversion formula to spherical coordinates

$$[F(r)\delta_{ij}+G(r)x_ix_j]\,dx^idx^j = \big(F(r)+r^2G(r)\big)\,dr^2+F(r)r^2d\Omega^2, \tag{4.18}$$

we find

$$h_{\mu\nu} = -B(r)dt^2+C(r)dr^2+A(r)r^2d\Omega^2, \tag{4.19}$$

where

$$B(r) = -\frac{2\kappa M}{3}\frac{1}{4\pi}\frac{e^{-mr}}{r}, \tag{4.20}$$

$$C(r) = -\frac{2\kappa M}{3}\frac{1}{4\pi}\frac{e^{-mr}}{r}\frac{1+mr}{m^2r^2}, \tag{4.21}$$

$$A(r) = \frac{\kappa M}{3}\frac{1}{4\pi}\frac{e^{-mr}}{r}\frac{1+mr+m^2r^2}{m^2r^2}. \tag{4.22}$$

$$\tag{4.23}$$

In the limit $r \ll 1/m$ these reduce to

$$B(r) = -\frac{2\kappa M}{3}\frac{1}{4\pi r}, \tag{4.24}$$

$$C(r) = -\frac{2\kappa M}{3}\frac{1}{4\pi m^2r^3}, \tag{4.25}$$

$$A(r) = \frac{\kappa M}{3}M\frac{1}{4\pi m^2r^3}. \tag{4.26}$$

$$\tag{4.27}$$

with corrections of order mr.

The metric as we have it is not in the right form to read off the Newtonian potential and light bending. To calculate the light bending, go back to Eq.(4.14) and notice that the $\frac{p_i p_j}{m^2}$ term in h_{ij} is pure gauge. Even though massive gravity has no gauge symmetry, its coupling to matter and light is still gauge invariant, so we can ignore this term. Thus our metric is gauge equivalent to the metric

$$\begin{aligned} h_{00}(x) &= \frac{2\kappa M}{3}\frac{1}{4\pi}\frac{e^{-mr}}{r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{\kappa M}{3}M\frac{1}{4\pi}\frac{e^{-mr}}{r}\delta_{ij}, \end{aligned} \tag{4.28}$$

Using the newtonian relations $2\kappa h_{00} = -2\phi$, $2\kappa h_{ij} = -2\psi\delta_{ij}$ and $\kappa^2 = 8\pi G$ we have, in the small mass limit,

$$\begin{aligned} \phi &= -\frac{4}{3}\frac{GM}{r}, \\ \psi &= -\frac{2}{3}\frac{GM}{r}\delta_{ij}. \end{aligned} \tag{4.29}$$

The magnitude of the light bending angle for light incident at impact parameter b is

$$\alpha = \frac{4GM}{b}, \tag{4.30}$$

the same value as in general relativity. If we were to try to make the newtonian potential agree with GR by scaling $G \to \frac{3}{4}G$, we'd have a theory with PPN parameter $\gamma = \frac{1}{2}$, and the lightbending would then change to $\frac{3GM}{b}$, off by 25 percent from GR. Thus linearlized massive gravity, even in the limit of zero mass, gives quantitatively different predictions from linearized GR. This is the vDVZ (van Dam, Veltman, Zakharov) discontinuity [3, 4].

5 Massive GR

What we want in a massive theory of gravity is some non-linear extension of the massive spin 2 theory. Unlike the case in GR, where the gauge invariance essentially constrains the extension to be Einstein gravity, the extension is not unique. No particularly compelling or natural ways are known. We expect that any such extension of GR will break the gauge symmetry, that is, the theory will not be generally covariant.

The first mindless attempt at such an extension is to deform GR by simply adding the Fierz-Pauli term to the full non-linear GR action:

$$S_{\rm m} = \frac{1}{2\kappa^2}\int d^Dx\ (\sqrt{-g}R) - \sqrt{-g^0}\frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \tag{5.1}$$

Here there are several subtlties. The lagrangian explicitly depends on a fixed metric $g^{(0)}_{\mu\nu}$, which we'll call the absolute metric. We have $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$ as before. Indices on $h_{\mu\nu}$ are raised and traced with the absolute metric. There is no way to introduce a mass term such as this using only the full metric $g_{\mu\nu}$, since tracing it with itself just gives a constant. The second, non-dynamical fixed metric is required to create the traces and contractions.

All we require of the deformation is that it reduce to the Fierz-Pauli term upon linearlization when the background is Minkowski. The way we have constructed it is not unique. For example, our Lagrangian has $\sqrt{-g^0}$ in front of the mass term rather than $\sqrt{-g}$, and indices on $h_{\mu\nu}$ are raised with the background metric rather than the full metric. If we were to use $\sqrt{-g}$, or raise indices with $g_{\mu\nu}$, it would not affect the linear theory, only the way it is extended non-linearly. We choose this way of doing it so that only terms with no derivatives are second order in $h_{\mu\nu}$. In reality, we imagine that there are undetermined terms cubic and higher order in $h_{\mu\nu}$, and without derivatives.

Varying $S_{\rm m}$ with respect to $g_{\mu\nu}$ we obtain the equations of motion

$$\sqrt{-g}(R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu})+\sqrt{-g^{(0)}}\frac{m^2}{2}\left(h^{\mu\nu}-hg^{(0)\mu\nu}\right)=0, \tag{5.2}$$

Indices on $R_{\mu\nu}$ and $G_{\mu\nu}$ are raised with the full metric, and those on $h_{\mu\nu}$ with the absolute metric. We see that if the absolute metric $g^{(0)}_{\mu\nu}$ satisfies the Einstein equations, then $g_{\mu\nu}=g^{(0)}_{\mu\nu}$, i.e. $h_{\mu\nu}=0$, is a solution. When dealing with massive gravity, there can be, in a sense, two different absolute structures. On the one hand, there is the absolute metric, the structure which breaks explicitly the diffeomorphism invariance. On the other hand, there is the background metric, which is a solution to the full non-linear equations, about which we can expand the action. Often, the solution metric we are expanding around will be the same as the absolute metric, but if we were expanding around a different solution, say a black hole, there would be two distinct structures, namely the black hole solution metric and the flat absolute metric. However, if when adding matter to the theory we agree to use only minimal coupling to the metric $g_{\mu\nu}$, then the absolute metric does not directly influence the matter. It is the geodesics and lengths as measured by the solution metric that we care about. If we have a solution metric, we cannot perform a diffeomorphism on it to obtain a second solution to the same theory, as we can in GR. What we can obtain, however, is a solution to a different massive gravity theory, one whose absolute metric is related to the original absolute metric by the same diffeomorphism.

Taking the second variation of $S_{\rm m}$ about the flat space solution, and rescaling $h_{\mu\nu}\rightarrow 2\kappa h_{\mu\nu}$, we obtain exactly the massive spin two lagrangian, $S_{\rm m\ linear}$.

5.1 Spherical solution, breakdown of linearity

We now specialize to four dimensions, and attempt to find spherically symmetric solutions to the equations of motion 5.2, in the case where the absolute metric is flat Minkowski,

$$g^{(0)}_{\mu\nu}=-dt^2+dr^2+r^2d\Omega^2.$$

The most general spherically symmetric static metric can be written

$$g_{\mu\nu} = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2. \tag{5.3}$$

Plugging this ansatz into the equations of motion, we get the following from the tt equation, rr equation and $\theta\theta$ equation (which is the same as the $\phi\phi$ equation by spherical symmetry) respectively,

$$4BC^2m^2r^2A^3 + \left(2B(C-3)C^2m^2r^2 - 4\sqrt{A^2BC}\left(C - rC'\right)\right)A^2$$

$$+2\sqrt{A^2BC}\left(2C^2 - 2r\left(3A' + rA''\right)C + r^2A'C'\right)A + C\sqrt{A^2BC}r^2\left(A'\right)^2 = 0,$$

$$\frac{4\left(B + rB'\right)A^2 + \left(2r^2A'B' - 4B\left(C - rA'\right)\right)A + Br^2\left(A'\right)^2}{A^2BC^2r^2} - \frac{2(2A+B-3)m^2}{\sqrt{A^2BC}} = 0$$

$$-2B^2C^2m^2rA^4 - 2B^2C^2(B+C-3)m^2rA^3$$

$$-\sqrt{A^2BC}\left(2C'B^2 + \left(rB'C' - 2C\left(B' + rB''\right)\right)B + Cr\left(B'\right)^2\right)A^2$$

$$+B\sqrt{A^2BC}\left(CrA'B' + B\left(4CA' - rC'A' + 2CrA''\right)\right)A - B^2C\sqrt{A^2BC}r\left(A'\right)^2 = 0.$$

In the massless case, $A(r)$ could be removed by a coordinate gauge transformation, and the last equation was redundant– it was a consequence of the first two. With non-zero m, there is no diffeomorphism invariance, so no such coordinate change can be made, and the last equation is independent.

We proceed to do a linear expansion of these equations around the flat space solution

$$B_0(r) = 1, \quad C_0(r) = 1, \quad A_0(r) = 1. \tag{5.4}$$

We do this by the usual method of linearizing a non-linear differential equation about a solution. We introduce the expansion

$$\begin{aligned} B(r) &= B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots, \\ C(r) &= C_0(r) + \epsilon C_1(r) + \epsilon^2 C_2(r) + \cdots, \\ A(r) &= A_0(r) + \epsilon A_1(r) + \epsilon^2 A_2(r) + \cdots, \end{aligned} \tag{5.5}$$

plugging into the equations of motion and collecting like powers of ϵ. The $\mathcal{O}(0)$ part gives $0 = 0$ because B_0, C_0, A_0 are solutions to the full non-linear equations. At each higher order in epsilon we will obtain a linear equation that lets us solve for the next term. At $\mathcal{O}(\epsilon)$ we obtain

$$2\left(m^2r^2 - 1\right)A_1 + \left(m^2r^2 + 2\right)C_1 + 2r\left(-3A_1' + C_1' - rA_1''\right) = 0,$$

$$-\frac{1}{2}B_1m^2 + \left(\frac{1}{r^2} - m^2\right)A_1 + \frac{r\left(A_1' + B_1'\right) - C_1}{r^2} = 0,$$

$$rA_1m^2 + rB_1m^2 + rC_1m^2 - 2A_1' - B_1' + C_1' - rA_1'' - rB_1'' = 0.$$

One way to solve these equations is as follows. Algebraically solve them simultaneously for A_1, A_1', A_1'' in terms of B_1's and C_1's and their derivatives. Then set

$\frac{d}{dr}A_1 = A_1'$ and $\frac{d}{dr}A' = A''$. Solve these two equations for C_1 and C_1' in terms of B_1's its derivatives. Then set $\frac{d}{dr}C_1 = C_1'$, and what you have is

$$-3rB_1m^2 + 6B_1' + 3rB_1'' = 0. \tag{5.6}$$

There are two constants in the solution, one is left arbitrary and the other must be sent to zero to prevent the solutions from blowing up at infinity. We then recursively determine C_1 and A_1. Thus the whole solution is determined by two pieces of inital data. Naively, it's a second order equation in A_1 and B_1, first order in C_1 and we might think this requires 5 initial conditions, but in fact it is a degenerate system, and there are second class constraints bringing the required initial data to 2.

The solution is

$$\begin{aligned} B_1(r) &= -\frac{8GM}{3}\frac{e^{-mr}}{r}, && (5.7)\\ C_1(r) &= -\frac{8GM}{3}\frac{e^{-mr}}{r}\frac{1+mr}{m^2r^2}, && (5.8)\\ A_1(r) &= \frac{4GM}{3}\frac{e^{-mr}}{r}\frac{1+mr+m^2r^2}{m^2r^2}. && (5.9)\\ & && (5.10)\end{aligned}$$

where we have chosen the integration constant so that we agree with the solution obtained from the green's function.

We can now proceed to $\mathcal{O}(\epsilon^2)$. Going through the same procedure, we find for the solution, when $1/r \gg m$,

$$\begin{aligned} B(r) - 1 &= -\frac{8}{3}\frac{GM}{r}\left(1 - \frac{1}{6}\frac{GM}{m^4r^5} + \cdots\right), && (5.11)\\ C(r) - 1 &= -\frac{8}{3}\frac{GM}{m^2r^3}\left(1 - 14\frac{GM}{m^4r^5} + \cdots\right), && (5.12)\\ A(r) - 1 &= \frac{4}{3}\frac{GM}{4\pi m^2r^3}\left(1 - 4\frac{GM}{m^4r^5} + \cdots\right). && (5.13)\\ & && (5.14)\end{aligned}$$

The dots represent higher powers in the non-linearity ϵ. We see that the the non-linearlity expansion is an expansion in the parameter r_v/r, where

$$r_V \equiv \left(\frac{r_S}{m^4}\right)^{1/5}, \tag{5.15}$$

is known as the Vainshtein radius. As the mass m approaches 0, r_V grows, and hence the radius beyond which the solution can be trusted gets pushed out to infinity. This particular perturbation expansion breaks down, and says nothing about the true non-linear behavior of massive gravity in the massless limit. Thus there is reason to hope that the vDVZ discontinuity is merely an artifact of linear perturbation theory, and the the true non-linear solutions show a smooth limit [8, 9, 10].

One might hope that a smooth limit could be seen by setting up an expansion in the mass m^2. We take a solution to the massless equations (the ordinary Schwartzchild solution), B_0, C_0, A_0, and then plug in an expansion

$$\begin{aligned} B(r) &= B_0(r) + m^2 B_1(r) + m^4 B_2(r) + \cdots , \\ C(r) &= C_0(r) + m^2 C_1(r) + m^4 C_2(r) + \cdots , \\ A(r) &= A_0(r) + m^2 A_1(r) + m^4 A_2(r) + \cdots , \end{aligned} \tag{5.16}$$

into the equations of motion, then collect powers of m. The equation we obtain at $\mathcal{O}(m^2)$ for the first correction to Schwartzschild is non-linear. It is quadratic in the variables, so working with this expansion is much more difficult than working with the linearlized expansion. It is not clear whether this expansion actually approximates a massive solution which approaches the massless one in the massless limit. In particular, there are issues with whether the solutions match on the exponentially decaying solutions correctly at infinity [14].

6 Stückelberg trick, decoupling

Here we'll see explicitly how the correct massless limit of massive gravity is not massless gravity, but rather massless gravity plus a scalar field which couples to the trace of the energy momentum tensor. Taking $m \to 0$ in $S_{\text{m linear}}$ is not a smooth limit. In particular, a gauge symmetry appears in the this limit and degrees of freedom are disappearing, so it might be expected that the limit is not smooth. The trick is to introduce gauge symmetry into the massive theory, in such a way that a limit can be taken in which no degrees of freedom are gained or lost in the limit.

6.1 Vector example

As a warm-up example, consider the theory of a massive photon coupled to a conserved source,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu. \tag{6.1}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.2}$$

The mass term breaks the would-be gauge invariance, $\delta A_\mu = \partial_\mu \Lambda$. As it stands, the limit $m \to 0$ is not a smooth limit, because a degree of freedom is lost. The Stückelberg trick consists of introducing a new scalar field ϕ, in such a way that the new action has gauge symmetry but is still dynamically equivalent to the original action. It will expose a different $m \to 0$ limit which is smooth, and in which no degrees of freedom are lost.

We introduce a field, ϕ, by making the replacement

$$A_\mu \to A_\mu + \partial_\mu \phi, \tag{6.3}$$

following the pattern of the gauge symmetry we want to introduce [16]. $F_{\mu\nu}$ is invariant under this replacement, since the replacement looks like a gauge transformation and $F_{\mu\nu}$ is gauge invariant. All that changes is the mass term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2(A_\mu + \partial_\mu\phi)^2 + A_\mu J^\mu. \tag{6.4}$$

The source term is unaffected because the change vanishes upon integration by parts and using conservation of the source. The action now has the gauge symmetry

$$\delta A_\mu = \partial_\mu\Lambda, \quad \delta\phi = -\Lambda, \tag{6.5}$$

and by fixing the gauge $\phi = 0$, a gauge condition for which it is permissible to substitute back into the action, we recover the original massive lagrangian.

We see from the above that ϕ has a kinetic term, in addition to cross terms. Rescaling $\phi \to \frac{1}{m}\phi$ in order to normalize the kinetic term, we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2A_\mu A^\mu - mA_\mu\partial^\mu\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu, \tag{6.6}$$

and the gauge symmetry is

$$\delta A_\mu = \partial_\mu\Lambda, \quad \delta\phi = -m\Lambda. \tag{6.7}$$

There is now a smooth $m \to 0$ limit. The *form* of the gauge symmetry changes in this limit, since ϕ loses its transformation, but the total *amount* of gauge symmetry in the action is the same before and after the limit, i.e. one gauge parameter. The lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu, \tag{6.8}$$

and the gauge symmetry is

$$\delta A_\mu = \partial_\mu\Lambda, \quad \delta\phi = 0. \tag{6.9}$$

The vector decouples from the scalar, and we are left with a massless gauge vector interacting with the source, as well as a completely decoupled free scalar. This $m \to 0$ limit is a different limit than the non-smooth limit we would have by taking $m \to 0$ straight away. We have scaled $\phi \to \frac{1}{m}\phi$ in order to canonically normalize the scalar kinetic term, so we are actually using a new scalar $\phi_{new} = m\phi_{old}$ which does not scale with m, so the smooth limit we are taking is to scale the old scalar degree of freedom up as we scale m down, in such a way that the new scalar degree of freedom remains constant.

The Stükelberg trick is a terrific illustration of the fact that gauge symmetry is a complete sham. It represents nothing more than a redundancy of description. We see that we can take any old theory and make it a gauge theory by introducing redundant variables. Similarly, given any gauge theory, we can always eliminate the gauge symmetry by eliminating the redundant degrees of freedom. The catch is that removing the redundancy is not always a smart thing to do. For example, in Maxwell

EM it is impossible to remove the redundancy and at the same time preserve manifest lorentz invariance and locality. Of course, the theory with gauge redundancy removed is still equivalent to Maxwell EM, so it is still lorentz invariant and local, it's just not manifestly so. With this Stükelberg trick, we are adding and removing extra gauge symmetry in a rather simple way, which happens to preserves manifest lorentz invariance and locality.

6.2 Filtering

As an aside, return to the lagrangian (6.6), before the $m \to 0$ limit. The ϕ equation of motion is

$$\Box\phi + m\partial \cdot A = 0. \tag{6.10}$$

Imagine integrating out the ϕ field in a path integral. Solving the equation of motion,

$$\phi = -\frac{m}{\Box}\partial \cdot A, \tag{6.11}$$

and plugging back into the action, we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\left(1 - \frac{m}{\Box}\right)F^{\mu\nu} + A_\mu J^\mu, \tag{6.12}$$

where we have used

$$F_{\mu\nu}\frac{1}{\Box}F^{\mu\nu} = -2A_\mu\frac{1}{\Box}A^\mu - 2\partial \cdot A\frac{1}{\Box}\partial \cdot A,$$

arrived at after much integration by parts. (6.12) is now a manifestly gauge invariant action lagrangian for a massive vector, which is non-local, the non-locality taking into account the longitudinal mode. The equation of motion is

$$\left(1 - \frac{m}{\Box}\right)\partial_\mu F^{\mu\nu} = -J^\nu. \tag{6.13}$$

This is simply Maxwell EM as seen thorough a high-pass filter, where m is the filter scale.

6.3 Massive Graviton Stükelberg

Now consider massive gravity,

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h_{\mu\nu}T^{\mu\nu}. \tag{6.14}$$

We want to preserve the gauge symmetry $\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ present in the $m = 0$ case, so we introduce a Stückelberg field patterned after the gauge symmetry,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu. \tag{6.15}$$

The $\mathcal{L}_{m=0}$ term remains invariant, the source term doesn't change due to conservation of $T^{\mu\nu}$ so all that changes is the mass term,

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} - 2m^2\left(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu\right) + \kappa h_{\mu\nu}T^{\mu\nu}.$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.16}$$

There is now a gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta A_\mu = -\xi_\mu, \tag{6.17}$$

and fixing the gauge $\xi_\mu = 0$ recovers the massive gravity action. At this point, we might consider scaling $A_\mu \to \frac{1}{m}A_\mu$ to normalize the vector kinetic term, then take the $m \to 0$ limit. The gauge symmetry for the vector would become $\delta A_\mu = -m\xi_\mu \to 0$, so we seem to have the same situation as in the massive photon example above, namely that the gauge symmetry changes but the amount doesn't, and that this should be a smooth limit. However this is not the case. Once m reaches zero, a new gauge symmetry appears, namely the usual gauge invariance of the kinetic term F^2, which has it's only scalar parameter Λ, $\delta A_\mu = \partial_\mu \Lambda$. So at this point, $m \to 0$ is still not a smooth limit. The number of degrees of freedom is still not conserved.

We have to go one step further and make explicit the scalar gauge symmetry that reappears, by introducing another Stückelberg field ϕ patterned after it,

$$A_\mu \to A_\mu + \partial_\mu \phi. \tag{6.18}$$

$$\begin{aligned}\mathcal{L} = \mathcal{L}_{m=0} \; &- \; \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} \\ &- \; 2m^2\left(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu\right) - 2m^2\left(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi\right) + \kappa h_{\mu\nu}T^{\mu\nu}.\end{aligned}$$

There are now two gauge symmetries

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta A_\mu = -\xi_\mu \tag{6.19}$$

$$\delta A_\mu = \partial_\mu \Lambda, \quad \delta\phi = -\Lambda. \tag{6.20}$$

By fixing the gauge $\phi = 0$ we recover the previous lagrangian.

We now rescale $A_\mu \to \frac{1}{\sqrt{2}m}A_\mu$, $\phi \to \frac{1}{m^2}\phi$, under which the gauge transformations become

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta A_\mu = -\sqrt{2}m\xi_\mu \tag{6.21}$$

$$\delta A_\mu = \sqrt{2}\partial_\mu \Lambda, \quad \delta\phi = -m\Lambda, \tag{6.22}$$

where we have absorbed one factor on m into the gauge parameter Λ. The $m \to 0$ limit is now smooth, no degrees of freedom are lost or gained. No new gauge gauge symmetry appears in the limit, and none is lost (the fact that m was absorbed into Λ does not mean that the gauge transformation actually vanishes in this limit, only that the gauge parameter must be made to grow, i.e. the gauge symmetry is still there). The theory now takes the form

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\left(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi\right) + \kappa h_{\mu\nu}T^{\mu\nu}, \tag{6.23}$$

with the gauge transformations (another $\sqrt{2}$ being absorbed into Λ),

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \quad \delta A_\mu = 0 \tag{6.24}$$
$$\delta A_\mu = \partial_\mu\Lambda, \quad \delta\phi = 0. \tag{6.25}$$

This is the smooth massless limit of massive gravity; a scalar tensor vector theory where the vector is completely decoupled but the scalar is kinetically mixed with the tensor. We can unmix them, at the expense of the minimal coupling to $T^{\mu\nu}$, by a field redefinition. Consider the change $h_{\mu\nu} \to h_{\mu\nu} + \phi\eta_{\mu\nu}$, the linearlization of a conformal transformation. The change in the massless spin-2 part is

$$\Delta\mathcal{L}_{m=0} = (D-2)\left[\partial_\mu\phi\partial^\mu h + \phi\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1}{2}(D-1)\partial_\mu\phi\partial^\mu\phi\right]. \tag{6.26}$$

This is simply the linearlization of the effect of a conformal transformation on the Einstein hilbert action.

By first scaling $\phi \to \frac{D-2}{2}\phi$, and then doing the above transformation, we will arrange to cancel all the off-diagonal $h\phi$ terms, trading them in for a ϕ kinetic term. Scaling $\phi \to \frac{1}{\sqrt{(D-1)(D-2)}}\phi$ then normalizes the kinetic term, leaving

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \kappa h_{\mu\nu}T^{\mu\nu} + \frac{1}{\sqrt{(D-1)(D-2)}}\kappa\phi T. \tag{6.27}$$

The theory is now in diagonal form, and we see explicitly that the 5 degrees of freedom of the massive graviton have, in the massless limit, become the two polarizations of a massless graviton coupled to the source, the two polarizations of a completely decoupled massless vector, and the single polarization of a massless scalar coupled with gravitational strength to the trace T of the energy momentum tensor.

We have exposed the origin of the vDVZ discontinuity. The extra scalar degree of freedom, which couples to the trace of the stress tensor, does not affect the bending of light (for which $T = 0$), but it does affect the Newtonian potential. This effect exactly accounts for the discrepancy between the massless limit of massive gravity and massless gravity.

As a side note, one can see from this Stückelberg trick that the Fiertz-Pauli form for the graviton mass is the correct one. Any deviation from this form, and the Stückelberg scalar will acquire a kinetic term with four derivatives, indicating extra pathological degrees of freedom. The Fiertz-Pauli coefficients are needed to exactly cancel these terms.

6.4 More general backgrounds

Consider again our massive gravity action, generalized to include a cosmological constant,

$$S_{\rm m} = \frac{1}{2\kappa^2}\int d^Dx\ \sqrt{-g}(R-2\Lambda) - \sqrt{-g^0}\frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu}-h^2). \tag{6.28}$$

Let's expand to quadratic order order about $h_{\mu\nu}=0$. We have

$$\begin{aligned} &\delta^2\left[\sqrt{-g}(R-2\Lambda) - \sqrt{-g^0}\frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu}-h^2)\right] \\ = \ &\sqrt{|g|}\left[-\frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{2}\nabla_\mu h\nabla^\mu h \right. \\ &\left. +\left(\Lambda-\frac{1}{2}R\right)\left(h^{\mu\nu}h_{\mu\nu}-\frac{1}{2}h^2\right) + 2R^{\mu\nu}\left(h_\mu^{\ \alpha}h_{\nu\alpha}-\frac{1}{2}h_{\mu\nu}h\right) - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu}-h^2)\right] + (\text{total } d). \end{aligned}$$

where after the equal sign all $g's$ are actually background metrics, and all covariant derivatives and contractions are with respect to the background metric. In manipulating this, the following expression may be useful to rearrange the second term

$$\nabla_\alpha h_{\mu\nu}\nabla^\mu h^{\alpha\nu} = \nabla_\mu h^{\mu\nu}\nabla^\alpha h_{\alpha\nu} - R_{\lambda\alpha}h^{\lambda\nu}h^\alpha_{\ \nu} + R_{\nu\lambda\alpha\mu}h^{\alpha\nu}h^{\mu\lambda} + (\text{total } \nabla). \tag{6.29}$$

As it stands, the action above is only expanded around a solution if the background metric satisfies einstein's equations $G_{\mu\nu}+\Lambda g_{\mu\nu}=0$. This implies (for $d\neq 2$)

$$R_{\mu\nu} = \frac{R}{d}g_{\mu\nu}, \quad \Lambda = \left(\frac{d-2}{2d}\right)R. \tag{6.30}$$

Using this, we have the massive gravity action at linear order,

$$\begin{aligned} S_{\rm m\ linear} \ = \ &\sqrt{-g}\left[-\frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{2}\nabla_\mu h\nabla^\mu h \right. \\ &\left. +\frac{R}{d}\left(h^{\mu\nu}h_{\mu\nu}-\frac{1}{2}h^2\right) - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu}-h^2)\right] + (\text{total } d). \end{aligned}$$

Notice the term, proportional to R, that kind of looks like a mass term, but not quite. There's some very interesting representation theory behind this, and a long discussion about what it means for a particle to be "massless" in a curved space time. It is commonly taken to mean any of three things: the action has a gauge symmetry, the dS/AdS representation the particle is in approaches a massless representation as the algebra contracts to the Poincare algebra, or waves propagate strictly along the light cones. These things all happen to be the same in flat space, but can be different in dS/AdS space [17].

6.5 Absence of vDVZ discontinuity in AdS/dS

Here we will see that the vDVZ discontinuity is absent in AdS space and dS space [5, 6, 7] . We have massive gravity on a curved space,

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu}-h^2) + \frac{1}{2}h_{\mu\nu}T^{\mu\nu}. \tag{6.31}$$

where we have omitted the overall $\sqrt{-g}$. The massless part has the gauge symmetry $\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ present in the $m = 0$ case, so as before, we introduce a Stückelberg field patterned after it,

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu A_\nu + \nabla_\nu A_\mu. \tag{6.32}$$

The $\mathcal{L}_{m=0}$ term remains invariant, the source term doesn't change due to covariant conservation of $T^{\mu\nu}$, so all that changes is the mass term,

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \\ &- \frac{1}{2} m^2 F_{\mu\nu} F^{\mu\nu} + \frac{2}{d} m^2 R A^\mu A_\mu - 2m^2 \left(h_{\mu\nu} \nabla^\mu A^\nu - h \nabla_\mu A^\mu \right) + \frac{1}{2} h_{\mu\nu} T^{\mu\nu}, \end{aligned} \tag{6.33}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{6.34}$$

and we have used the relation

$$\nabla_\mu A_\nu \nabla^\nu A^\mu = (\nabla_\mu A^\mu)^2 - R_{\mu\nu} A^\mu A^\nu \tag{6.35}$$

to see that there is now a term that looks like a mass for the vector, proportional to the background curvature. There is now a gauge symmetry

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad \delta A_\mu = -\xi_\mu, \tag{6.36}$$

and fixing the gauge $\xi_\mu = 0$ recovers the massive gravity action.

We can then go to canonical normalization for the vector by taking $A_\mu \to \frac{1}{\sqrt{2}m} A_\mu$. Then we notice that we can smoothly take the $m \to 0$ limit, without the need to introduce the second Stückelberg field ϕ. This is because a mass term for the vector is present in this limit, so no degrees of freedom are lost, and no gauge invariance gained, as was the case in flat space. Thus our action is

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{R}{d} A^\mu A_\mu + \frac{1}{2} h_{\mu\nu} T^{\mu\nu}. \tag{6.37}$$

The massive vector completely decouples, so there is no vDVZ discontinuity. Notice that the vector is a tachyon in dS space but healthy in AdS.

7 Non-linear Stükelberg

We now want to extend the Stükelberg trick to full non-linear order, which will be useful in elucidating the breakdown in the linear expansion as due to strong coupling of the Stükelberg scalar. It will also tell us about quantum corrections and where we can expect them to become important.

7.1 Spin one example

Consider a non-abelian $SU(N)$ gauge theory, where we've added a non-gauge invariant mass term for the gauge bosons,

$$\mathcal{L} = \frac{1}{2g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{g^2}\mathrm{Tr}A_\mu A^\mu. \tag{7.1}$$

As usual, the gauge fields take values in the lie algebra

$$A_\mu = -igA^a_\mu T_a.$$

$$[T_a, T_b] = if_{ab}{}^c T_c, \quad Tr(T_a T_b) = \frac{1}{2}\delta_{ab}.$$

The field strength is,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$
$$F_{\mu\nu} = -igF^a_{\mu\nu}T_a,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{bc}{}^a A^b_\mu A^c_\nu.$$

In the absence of the mass term, the action is invariant under the gauge transformations (in matrix notation)

$$A_\mu \to RA_\mu R^\dagger + R\partial_\mu R^\dagger,$$

where

$$R = e^{-i\alpha^a T_a} \in SU(N),$$

and $\alpha^a(x)$ are gauge parameters. This reads infinitesimally

$$\delta A^a_\mu = \frac{1}{g}\partial_\mu \alpha^a + f_{bc}{}^a A^b_\mu \alpha^c.$$

The field strength transforms covariantly

$$F_{\mu\nu} \to RF_{\mu\nu}R^\dagger,$$

which reads infinitesimally

$$\delta F^a_{\mu\nu} = f_{bc}{}^a \alpha^b F^c_{\mu\nu}.$$

We want to add fields, patterned after this gauge symmetry, so we make the replacement

$$A_\mu \to UA_\mu U^\dagger + U\partial_\mu U^\dagger, \tag{7.2}$$

where

$$U = e^{-i\pi^a T_a} \in SU(N),$$

and the $\pi^a(x)$ are scalar goldstone fields. The gauge kinetic term is invariant under this replacement, since it is gauge invariant. The action now becomes gauge invariant under right gauge transformations,

$$A_\mu \to RA_\mu R^\dagger + R\partial_\mu R^\dagger, \quad U \to UR^\dagger. \tag{7.3}$$

(making the replacement $A_\mu \to U^\dagger A_\mu U - U\partial_\mu U^\dagger$ would have led to left gauge transformations.) The mass term becomes

$$\mathcal{L}_m \to -\frac{m^2}{g^2}\mathrm{Tr}D_\mu U^\dagger D^\mu U, \tag{7.4}$$

where

$$D_\mu U \equiv \partial_\mu U - UA_\mu$$

is a covariant derivative, which transforms covariantly under right gauge transformations,

$$D_\mu U \to (D_\mu U)R^\dagger. \tag{7.5}$$

We can go to the unitary gauge $U = 1$, and recover the massive vector action we started with.

The sigma model mass term is invariant under $SU(N)_L \times SU(N)_R$ global symmetry, $U \to LUR^\dagger$, of which the $SU(N)_R$ part is gauged. The $SU(N)$ subgroup $L = R$ is realized linearly, and the rest is realized non-linearly. It can be shown that the goldstones become strongly coupled at energies $\sim \frac{4\pi m}{g}$, and so there will be quantum corrections looking like

$$\frac{1}{16\pi^2}\mathrm{Tr}\left[D_\mu U^\dagger D^\mu\right]^2, \quad \frac{1}{16\pi^2}\mathrm{Tr}\left[D^2U^\dagger D^2 U\right], \ldots \tag{7.6}$$

which in unitary gauge look like

$$\frac{1}{16\pi^2}\mathrm{Tr}A^4, \quad \frac{1}{16\pi^2}\mathrm{Tr}(\partial A)^2, \ldots \tag{7.7}$$

Notice that this second operator modifies the gauge kinetic term in a non-gauge invariant way, and leads to ghosts/tachyons. However, its size is small enough that these are all pushed to the cutoff.

Another way to introduce the Stückelberg fields, which will be more like the way we do it in the gravity case, is to start with a Yang-Mills theory that has two gauge invariances, $SU(N)_L \times SU(N)_R$,

$$\mathcal{L} = \frac{1}{2g_L^2}\mathrm{Tr}F_{L\mu\nu}F_L^{\mu\nu} + \frac{1}{2g_R^2}\mathrm{Tr}F_{R\mu\nu}F_R^{\mu\nu} + \cdots \tag{7.8}$$

We now introduce a sigma model link field $U = e^{-i\pi^a T_a} \in SU(N)$, which transforms as

$$U \to LUR^\dagger, \tag{7.9}$$

and add a "hopping" term to the lagrangian,

$$\mathcal{L} = \frac{1}{2g_L^2}\mathrm{Tr}F_{L\mu\nu}F_L^{\mu\nu} + \frac{1}{2g_R^2}\mathrm{Tr}F_{R\mu\nu}F_R^{\mu\nu} - f^2\mathrm{Tr}D_\mu U^\dagger D^\mu U, \tag{7.10}$$

where the covariant derivative is

$$D_\mu U \equiv \partial_\mu U + A_L U - UA_R \tag{7.11}$$

and transforms homogeneously under $U \to LUR^\dagger$,

$$D_\mu U \to L(D_\mu U)R^\dagger. \tag{7.12}$$

We can think of the gauge fields as living on two different sites, L and R, and U as a link field that connects the two sites.

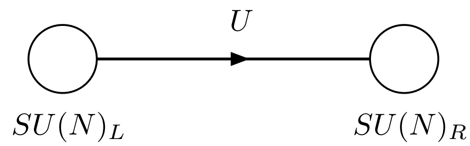

Figure 4: Site mnemonic for non-abelian Stückelberg fields.

We can go to unitary gauge where $U = 1$, where the only gauge symmetry is the $R = L$,

$$\mathcal{L} = \frac{1}{2g_L^2}\mathrm{Tr}F_{L\mu\nu}F_L^{\mu\nu} + \frac{1}{2g_R^2}\mathrm{Tr}F_{R\mu\nu}F_R^{\mu\nu} + f^2\mathrm{Tr}(A_L - A_R)^2. \tag{7.13}$$

Expanding the gauge fields over the generators, the mass term becomes $-\frac{1}{2}f^2(g_L A_L^a - g_R A_R^a)^2$, corresponding to the mass matrix

$$f^2 \begin{pmatrix} g_L^2 & -g_L g_R \\ -g_L g_R & g_R^2 \end{pmatrix}, \tag{7.14}$$

which has eigenvalues $0,\ f^2(g_L^2 + g_R^2)$. Thus the spectrum of the theory is one set of $N^2 - 1$ massless gauge bosons, and one set of vector bosons of mass $f\sqrt{g_L^2 + g_R^2}$.

Consider the limit $g_L \to 0$. The massless gauge bosons become all A_L and decouple, the massive ones become all A_R, and we are left with a theory of one set of massive gauge bosons with mass fg_R. Notice that this limit is not smooth. We are losing the massless vector degrees of freedom, and we are losing one set of gauge invariances. Taking this limit before going to unitary gauge, we have the theory of massive vector bosons with a single $SU(N)$ gauge symmetry, exactly the what we had after Stückelberg-ing the massive Yang-Mills theory.

7.2 Spin two

We now construct the gravitational analogue of the above, following [11]. We start with a collection of spacetimes, called sites, (all taken to be $\mathbb{R}^n$ for simplicity), labelled by $i, j, \ldots$. Each has its own coordinates x_i, x_j, etc. Each can have fields $\phi_i(x_i)$, $\phi_j(x_j)$, etc., which may be scalars, vectors, tensors, or whatever.

Each site has its own (active) general coordinate transformations GC_j. GC_j acts on the coordinates of j in the usual way, $x_j^\mu \to f_j^\mu(x_j)$. The functions f_j are of course smooth and invertible. A scalar $\phi(x_j)$ on site j transforms under GC_j by

$$\phi(x_j) \to \phi\left(f_j(x_j)\right). \tag{7.15}$$

In terms of function composition, this is just

$$\phi \to \phi \circ f_j. \tag{7.16}$$

The fields on site j do not transform under GC_i when $i \neq j$. Similarly, a vector field $a_{j\mu}(x_j)$ transforms under GC_j as

$$a_{j\mu}(x_j) \to \frac{\partial f_j^\alpha}{\partial x_j^\mu}(x_j) a_{j\alpha}(f(x_j)), \tag{7.17}$$

and so on for all other tensor fields.

We now introduce a link field Y_{ji}, which is a map from site i to site j. Hence it is a set of d fields on site i, but it transforms under both GC_i and GC_j,

$$Y_{ji} \to f_j^{-1} \circ Y_{ji} \circ f_i \tag{7.18}$$

$$Y_{ji}^\mu(x_i) \to (f_j^{-1})^\mu(Y_{ji}(f_i(x_i))). \tag{7.19}$$

Given scalar or co-vector fields on site j, we can now pull them back to site i using the link field Y. For example, given a scalar $\phi(x_j)$, vector $a_\mu(x_j)$ and a metric $g_{\mu\nu}(x_j)$, which transform in the usual way under GC_j and are invariant under GC_i, we can form the objects

$$\Phi(x_i) = \phi_j(Y_{ji}(x_i)) \tag{7.20}$$

$$A_\mu(x_i) = \frac{\partial Y^\alpha}{\partial x_i^\mu}(x_i) a_{j\alpha}(Y_{ji}(x_i)), \tag{7.21}$$

$$G_{\mu\nu}(x_i) = \frac{\partial Y^\alpha}{\partial x_i^\mu}(x_i)\frac{\partial Y^\beta}{\partial x_i^\nu}(x_i) g_{j\alpha\beta}(Y_{ji}(x_i)) \tag{7.22}$$

which transform as a scalar, vector and metric respectively under GC_i, and are invariant under GC_j.

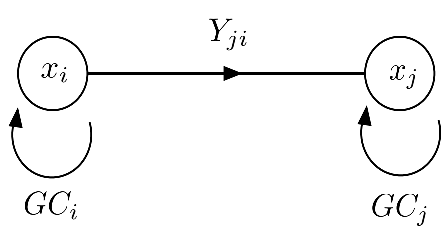

Figure 5: Site mnemonic for gravity fields.

7.3 Goldstone boson expansion

We can now expand Y about the identity

$$Y^\alpha(x) = x^\alpha + \pi^\alpha(x) \tag{7.23}$$

where here and in what follows we have dropped the ij indices on the Y and the i index on x to avoid notational clutter. All the arguments are x_i and all the stuff is happening on site i.

The metric $G_{\mu\nu}$, can be expanded as

$$\begin{aligned} G_{\mu\nu} &= \frac{\partial Y^\alpha(x)}{\partial x^\mu}\frac{\partial Y^\beta(x)}{\partial x^\nu} g^j_{\alpha\beta}(Y(x)) = \frac{\partial(x^\alpha+\pi^\alpha)}{\partial x^\mu}\frac{\partial(x^\beta+\pi^\beta)}{\partial x^\nu} g^j_{\alpha\beta}(x+\pi) \\ &= (\delta^\alpha_\mu + \partial_\mu\pi^\alpha)(\delta^\beta_\nu+\partial_\nu\pi^\beta)(g^j_{\alpha\beta} + \pi^\mu\partial_\mu g^j_{\alpha\beta} + \frac{1}{2}\pi^\mu\pi^\nu\partial_\mu\partial_\nu g^j_{\alpha\beta} + \cdots) \\ &= g^j_{\mu\nu} + \pi^\lambda\partial_\lambda g^j_{\mu\nu} + \partial_\mu\pi^\alpha g^j_{\alpha\nu} + \partial_\nu\pi^\alpha g^j_{\alpha\mu} + \frac{1}{2}\pi^\alpha\pi^\beta\partial_\alpha\partial_\beta g^j_{\mu\nu} \\ &\quad + \partial_\mu\pi^\alpha\partial_\nu\pi^\beta g^j_{\alpha\beta} + \partial_\mu\pi^\alpha\pi^\beta\partial_\beta g^j_{\alpha\nu} + \partial_\nu\pi^\alpha\pi^\beta\partial_\beta g^j_{\mu\alpha} + \cdots \end{aligned} \tag{7.24}$$

We can look at the transformation properties of g, G, Y and π, under infinitesimal general co-ordinate transformations generated by $f_i(x) = x + \xi_i(x)$ and $f_j(x) = x + \xi_j(x)$. The metrics on the sites transform as

$$\delta g^i_{\mu\nu} = \xi^\lambda_i\partial_\lambda g^i_{\mu\nu} + \partial_\mu\xi^\lambda_i g^i_{\lambda\nu} + \partial_\nu\xi^\lambda_i g^i_{\mu\lambda} \tag{7.25}$$
$$\delta g^j_{\mu\nu} = \xi^\lambda_j\partial_\lambda g^j_{\mu\nu} + \partial_\mu\xi^\lambda_j g^j_{\lambda\nu} + \partial_\nu\xi^\lambda_j g^j_{\mu\lambda} \tag{7.26}$$
$$\delta\sqrt{g_i} = \partial_\lambda\xi^\lambda_i\sqrt{g_i} + \xi^\lambda_i\partial_\lambda(\sqrt{g_i}) \tag{7.27}$$
$$\delta\sqrt{g_j} = \partial_\lambda\xi^\lambda_j\sqrt{g_j} + \xi^\lambda_j\partial_\lambda(\sqrt{g_j}) \tag{7.28}$$

The transformation laws of the goldstones come from the transformation of the link Y. Under GC$_i$:

$$\begin{aligned} Y(x)\to Y(x') &= x+\xi_i+\pi(x+\xi_i) \equiv x+\pi+\delta\pi \\ \Rightarrow \delta\pi^\mu &= \xi^\mu_i + \xi^\alpha_i\partial_\alpha\pi^\mu \end{aligned} \tag{7.29}$$

Under GC$_j$

$$\begin{aligned} Y &\to Y - \xi_j(Y) = x+\pi-\xi_j(x+\pi) \equiv x+\pi+\delta\pi \\ \Rightarrow \delta\pi^\mu &= -\xi^\mu_j(x+\pi) = -\xi^\mu_j - \pi^\alpha\partial_\alpha\xi^\mu_j - \frac{1}{2}\pi^\alpha\pi^\beta\partial_\alpha\partial_\beta\xi^\mu_j + \cdots \end{aligned} \tag{7.30}$$

So the goldstones transform under the two transformations as

$$\delta\pi^\mu = \xi^\mu_i + \xi^\beta_i\partial_\beta\pi^\mu - \xi^\mu_j - \pi^\beta\partial_\beta\xi^\mu_j - \frac{1}{2}\pi^\alpha\pi^\beta\partial_\alpha\partial_\beta\xi^\mu_j - \cdots \tag{7.31}$$

In the global symmetry limit , where the ξ's are constant, we have

$$\pi^\mu \to \pi^\mu + \xi^\nu_i\partial_\nu\pi^\mu + \xi_i - \xi_j \tag{7.32}$$

This is just a translation in x_i by ξ_i, together with a shift symmetry. Note that in this global limit the symmetry is Abelian.

$G_{\mu\nu}$ has the expected transformation law

$$\delta G_{\mu\nu} = \xi^\lambda_i\partial_\lambda G_{\mu\nu} + \partial_\mu\xi^\lambda_i G_{\lambda\nu} + \partial_\nu\xi^\lambda_i G_{\mu\lambda}. \tag{7.33}$$

$G_{\mu\nu}$ transforms like a tensor under GC$_i$ and is invariant under GC$_j$.

7.4 Hopping Action

We can now construct the action

$$S = S_{grav} + S_{mass} \tag{7.34}$$

where

$$S_{grav} = \frac{M_i^2}{2}\int d^4x_i\sqrt{-g_i}\left(R[g_i] - 2\Lambda_i + \cdots\right) + \frac{M_j^2}{2}\int d^4x_j\sqrt{-g_j}\left(R[g_j] - 2\Lambda_j + \cdots\right) \tag{7.35}$$

represents the action for gravitons on the sites,

$$S_{mass} = -\frac{M_i^2}{2}\frac{m^2}{4}\int d^4x_i\sqrt{-g_i}g_i^{\mu\nu}g_i^{\alpha\beta}\left(H_{\mu\alpha}H_{\nu\beta} - H_{\mu\nu}H_{\alpha\beta}\right) + \cdots \tag{7.36}$$

is the "hopping" action that will give one combination of gravitons a mass. Here,

$$H_{\mu\nu}(x_i) \equiv g_{i\mu\nu}(x_i) - \partial_\mu Y^\alpha(x_i)\partial_\nu Y^\beta(x_i)g_{j\alpha\beta}(Y(x_i)). \tag{7.37}$$

S_{grav} is trivially invariant under $\mathrm{GC}_i\times\mathrm{GC}_j$. S_{grav} is also invariant under $\mathrm{GC}_i\times\mathrm{GC}_j$, but in a way that involves fields on the two sites. We can go to a unitary gauge where Y=id and there is one manifest general coordinate invariance under which both g_i and g_j transform as tensors, namely the diagonal one $f_i = f_j$. In this gauge, we have $x_i = x_j$, $G_i^{\mu\nu} = g_j^{\mu\nu}$, $H_{\mu\nu} = g_{i\mu\nu} - g_{j\mu\nu}$, and all fields can be thought of as depending on x_i.

The theory is then seen to contain one massless graviton and one massive graviton. In the limit where we send $M_j \to \infty$, this massless graviton is all g_j and becomes non-dynamical, and we are left with our original action for a theory of a single massive graviton described by g_i, in a non-dynamical background geometry g_j.

In terms of the goldstone expansion, we have (where g_j is now just g, the non-dynamical background), and $h = G - g$.

$$\begin{aligned} H_{\mu\nu} \quad = \quad & h_{\mu\nu} + \pi^\lambda\partial_\lambda g^j_{\mu\nu} + \partial_\mu\pi^\alpha g^j_{\alpha\nu} + \partial_\nu\pi^\alpha g^j_{\alpha\mu} + \frac{1}{2}\pi^\alpha\pi^\beta\partial_\alpha\partial_\beta g^j_{\mu\nu} \\ & + \partial_\mu\pi^\alpha\partial_\nu\pi^\beta g^j_{\alpha\beta} + \partial_\mu\pi^\alpha\pi^\beta\partial_\beta g^j_{\alpha\nu} + \partial_\nu\pi^\alpha\pi^\beta\partial_\beta g^j_{\mu\alpha} + \cdots \end{aligned} \tag{7.38}$$

To linear order, the expansion reads

$$H_{\mu\nu} = h_{\mu\nu} + \nabla_\mu\pi_\nu + \nabla_\nu\pi_\mu, \tag{7.39}$$

where indices on π are lowered with the background metric. This is exactly the goldstone substitution we made earlier in the linear case, patterned after the linear gauge symmetry. In the case where the background is flat, we have to all orders

$$H_{\mu\nu} = h_{\mu\nu} + \partial_\mu\pi_\nu + \partial_\nu\pi_\mu + \partial_\mu\pi^\alpha\partial_\nu\pi_\alpha. \tag{7.40}$$

This takes into account the full non-linear gauge transformation.

8 Goldstone expansion for massive gravity

Take the mass term when the background is Minkowksi

$$S_{mass} = -\frac{M_P^2}{2}\frac{m^2}{4}\int d^4x\eta^{\mu\nu}\eta^{\alpha\beta}\left(H_{\mu\alpha}H_{\nu\beta} - H_{\mu\nu}H_{\alpha\beta}\right) \tag{8.1}$$

then we make the replacement,

$$H_{\mu\nu} = h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + \partial_\mu A^\alpha \partial_\nu A_\alpha. \tag{8.2}$$

followed by another Stükelberg to make the $U(1)$ manifest,

$$A_\mu \to A_\mu + \partial_\mu \phi. \tag{8.3}$$

Once this is done, we scale all the variable $h_{\mu\nu} \to 2\kappa h_{\mu\nu},\ A_\mu \to 2\kappa A_\mu,\ \ \phi \to 2\kappa\phi$, corresponding to canonically normalizing the graviton kinetic term. Then we scale $A_\mu \to \frac{1}{\sqrt{2}m},\ \ \phi \to \frac{1}{m^2}\phi$, followed by the conformal transformation $h_{\mu\nu} \to h_{\mu\nu} + \phi\eta_{\mu\nu}$. This will diagonalize all the kinetic terms (except for hA cross terms proportional to m), and leave them all with canonical normalization (the ϕ kinetic term is left as $-3(\partial\phi)^2$ for convenience).

Expanding out the Fiertz-Pauli term in this way, we also get a whole slew of interaction terms, suppressed by various scales. We always assume $m < M_p$. The term suppressed by the smallest scale is the cubic scalar term, which is suppressed by the scale $\Lambda_5 = (M_p m^4)^{1/5}$,

$$\sim \frac{(\partial^2\phi)^3}{M_P m^4}. \tag{8.4}$$

The next highest scale is $\Lambda_4 = (M_p m^3)^{1/4}$, carried by a quartic scalar interaction,

$$\sim \frac{(\partial^2\phi)^4}{M_p^2 m^6}. \tag{8.5}$$

(terms $\sim \partial A(\partial^2\phi)^2$ would also carry this scale, but they all vanish. In fact, all term of the form $\sim \partial A(\partial^2\phi)^n$ vanish.) The next scale is $\Lambda_3 = (M_p m^2)^{1/3}$, which is carried by terms with other stuff besides the scalar

$$\sim \frac{(\partial A)^2\partial^2\phi}{M_p m^2},\quad \frac{(\partial A)^2(\partial^2\phi)^2}{M_p^2 m^4},\quad \frac{(h+\phi)(\partial^2\phi)^2}{M_p m^2} \tag{8.6}$$

8.1 Decoupling limit and breakdown of linearity

The lowest scale is Λ_5, so this is the cutoff of the effective field theory. To focus in on the cutoff scale, we take the limit

$$m \to 0,\quad M_p \to \infty,\quad \Lambda_5 \text{ fixed}. \tag{8.7}$$

All interaction terms (including the hA cross terms in the quadratic part) go to zero, except for the scalar cubic term responsible for the strong coupling. The lagrangian for the scalar is

$$\mathcal{L}_\phi = -3(\partial\phi)^2 + \frac{1}{\Lambda_5^5}\left[(\Box\phi)^3 - (\Box\phi)(\partial_\mu\partial_\nu\phi)^2\right] + \frac{1}{M_p}\phi T. \tag{8.8}$$

We can now understand the origin of the scale at which the linear expansion breaks down around heavy point sources. The goldstone scalar couples to the source through the trace, $\frac{1}{M_p}\phi T$. We do perturbation theory to find the classical value of ϕ around the source, using the three point vertex above. Each external source gets one power of $\frac{M}{M_p}$, each 3-point vertex gets one power of $\frac{1}{\Lambda^5}$, and the power of r is obtained by dimensional analysis. To linear order we have

$$\phi \sim \frac{M}{M_p}\frac{1}{r}. \tag{8.9}$$

In fact, any graph with n external currents will have $n-1$ three point vertices and will go like

$$\left(\frac{M}{M_p}\right)^n \frac{1}{\Lambda_5^{5(n-1)} r^{5(n-1)+1}}. \tag{8.10}$$

We see that each higher order is suppressed from the order before it by the factor

$$\frac{M}{M_p}\frac{1}{\Lambda^5 r^5}, \tag{8.11}$$

so that when $r < \left(\frac{M}{M_P}\right)^{1/5}\frac{1}{\Lambda_5} \equiv R_V$, the perturbation theory breaks down and non-linear effects become important. This is exactly the Vainstein radius found by directly calculating the second order correction.

$$R_V \sim \left(\frac{M}{M_P}\right)^{1/5}\frac{1}{\Lambda_5} \sim \left(\frac{GM}{m^4}\right)^{1/5}. \tag{8.12}$$

8.2 Ghosts

Following [12], let's consider the stability of the classical solution around a massive point source. We have a classical background $\Phi(r)$, which is a solution of the ϕ equation of motion. We expand the Lagrangian to quadratic order in the fluctuation $\varphi \equiv \phi - \Phi$. The result is schematically

$$\mathcal{L}_\varphi \sim -(\partial\varphi)^2 + \frac{(\partial^2\Phi)}{{\Lambda_5}^5}(\partial^2\varphi)^2. \tag{8.13}$$

There is a four-derivative contribution to the φ kinetic term. This signals the appearance of a ghost with an r-dependent mass

$$m^2_{\text{ghost}}(r) \sim \frac{{\Lambda_5}^5}{\partial^2\Phi(r)}\ . \tag{8.14}$$

We are working in an effective field theory with a UV cutoff Λ_5, therefore we should not worry until the mass of the ghost drops below Λ_5. This happens at the distance $r_{\rm ghost}$ where $\partial^2\Phi^c \sim {\Lambda_5}^3$. For a source of mass M, at distances $r \gg r_V$ the background field goes like $\Phi(r) \sim \frac{M}{M_P}\frac{1}{r}$, so

$$r_{\rm ghost} \sim \left(\frac{M}{M_P}\right)^{1/3}\frac{1}{\Lambda_5} \gg r_V \sim \left(\frac{M}{M_P}\right)^{1/5}\frac{1}{\Lambda_5}. \tag{8.15}$$

$r_{\rm ghost}$ is parametrically larger than the Vainshtein radius r_V. As we'll see in the next section, the distance $r_{\rm ghost}$ is the same distance at which quantum effects become important. Whatever UV completion takes over should cure the ghost instabilities that become present at this scale. We see already that we cannot even trust the classical solution up to the Vainstein radius. The best we can do is make predictions outside $r_{\rm ghost}$.

8.3 Sixth degree of freedom

A particularly nice way to study massive gravity is through the ADM Hamiltonian formalism. This has the advantage of explicitly displaying the degrees of freedom. A $3+1$ slicing of spacetime is chosen, and the ten components of the metric $g_{\mu\nu}$ are written in terms of the spatial metric g_{ij}, the lapse N_i and the shift N. The lapse and shift describe how to evolve the spatial metric from slice to slice.

In the case of GR, the lapse and shift appear in the action linearly and without time derivatives, so they act as lagrange multipliers that enforce constraints among the g_{ij}. 10 metric components, minus 4 constraints, minus 4 lagrange multipliers leaves two degrees of freedom. The non-linear theory contains the same number of degrees of freedom as the linearlized theory.

In the case of massive gravity, the Fiertz-Pauli term brings in contributions to the action that are quadratic in the lapse and shift (but still free of time derivatives), so they no longer serve as lagrange multipliers but rather as auxiliary fields. Their equations of motion only serve to determined their values, they do not fix additional constraints. Thus we have 10 metric components, minus 4 auxiliary fields, leaving 6 degrees of freedom for massive gravity. The linearlized theory only had five degrees of freedom, and we have here the situation where the non-linear theory contains more degrees of freedom than the linear theory [15].

There is no way to eliminate this extra degree of freedom by adding terms higher order in $h_{\mu\nu}$. Around flat space, this degree of freedom is not excited, but around non-trivial backgrounds it becomes active, and is in fact responsible for the ghost [12].

8.4 Quantum effective theory

Quantum mechanically we will generate all operators compatible with the fake gauge symmetries, suppressed by the appropriate power of Λ_5. It can be shown that the shift symmetry guarantees that the leading operators are of the form

$$\sim \frac{\partial^q (\partial^2 \phi^c)^p}{\Lambda_5^{3p+q-4}}. \tag{8.16}$$

We can go back to the original normalization for the fields by scaling $\phi \to m^2 M_p \phi$ and recall that $\partial_\mu \partial_\nu \phi$ always comes from an $h_{\mu\nu}$ to find that in unitary gauge, we have operators of the form

$$c_{p,q} \partial^q h^p \tag{8.17}$$

where the coefficients $c_{p,q}$ go like

$$c_{p,q} \sim \Lambda_5^{-3p-q+4} M_p^p m^{2p} = \left(m^{16-4q-2p} M_p^{2p-q+4}\right)^{1/5}. \tag{8.18}$$

Notice that the term with $p = 2,\ q = 0$ is a mass term that ruins the Fiertz-Pauli tuning, but its coefficient is small enough that ghost/tachyons are postponed to the cutoff. Thus in unitary gauge, there is a natural effective field theory with the action

$$\mathcal{L} = \frac{M_p}{2} \left[\sqrt{-g} R - \frac{m^2}{4} (h_{\mu\nu}^2 - h^2) \right] + \sum_{p,q} c_{p,q} \partial^q h^p \tag{8.19}$$

with a cutoff $\Lambda_5 = (m^4 M_p)^{1/5}$

If we try to take into account the effect that the quantum operators have on the solution around a heavy source, we should include diagrams with interactions drawn from n point vertices of the form $\partial^q (\partial^2 \phi_c)^n / \Lambda^{3n+q-4}$ which contribute a factor of $1/\Lambda^{3n+q-4}$. The contribution to ϕ from a diagram with a single such vertex is

$$\phi^{(n,q)} \sim \left(\frac{M}{M_P}\right)^{n-1} \frac{1}{\Lambda_5^{3n+q-4}} \frac{1}{r^{3n+q-3}} \tag{8.20}$$

The distance r_n at which this n'th order contribution to ϕ becomes comparable to the lowest order contribution, $\phi \sim \frac{M}{M_p} \frac{1}{r}$, is then

$$r_{n,q} \sim \left(\frac{M}{M_{Pl}}\right)^{\frac{n-2}{3n+q-4}} \frac{1}{\Lambda_5} \tag{8.21}$$

This distance increases with n, and asymptotes to

$$r_* \sim \left(\frac{M}{M_{Pl}}\right)^{1/3} \frac{1}{\Lambda_5}. \tag{8.22}$$

Thus we cannot trust the classical solution at distances below r_*, since quantum operators become important there. This distance is parametrically larger than the Vainshtein radius, where classical non-linearities become important. Unlike the case in GR, there is no intermediate regime where the linear approximation breaks down but quantum effects are still small, so there is no sense in which a non-linear solution to massive gravity should be trusted for making real predictions. Notice also that it is the higher dimension operators that become important first, so there is no hope of finding the leading quantum corrections. The theory transitions directly from the linear classical regime to the full quantum regime.

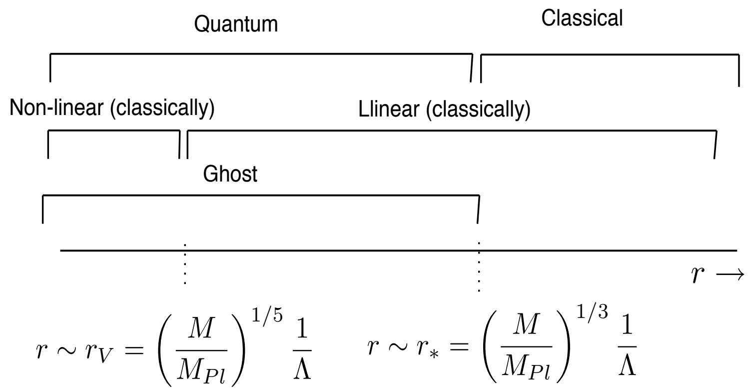

Figure 6: Regimes for massive gravity with cutoff $\Lambda_5 = (M_p m^4)^{1/5}$.

8.5 Adding interactions to raise the cutoff

With the Fiertz-Pauli mass term, the strong coupling cutoff was set by the cubic scalar self coupling $\sim \frac{(\partial^2\phi)^3}{\Lambda_5^5}$. We might try to change the theory by adding terms cubic and higher in $H_{\mu\nu}$. These will generate many more interactions, but the strongest coupling will always be the lowest power scalar self coupling, of the form $(\partial^2\phi)^n$.

We can arrange to cancel all of the scalar self couplings by adding appropriate higher order terms. We follow the procedure outlined in [12]. Start with the Fierz-Pauli mass term $\mathcal{L}_2 = ([H^2] - [H]^2)$. We use the notation

$$\begin{aligned} [H] &= H_\mu{}^\mu, && (8.23) \\ \left[H^2\right] &= H_{\mu\nu}H^{\nu\mu}, && (8.24) \\ \left[H^3\right] &= H_{\mu\nu}H^\nu{}_\lambda H^{\lambda\mu}, && (8.25) \\ &\vdots && (8.26) \end{aligned}$$

We are interested only in scalar self interactions, so we make the replacement

$$H_{\mu\nu} = h_{\mu\nu} + 2\,\partial_\mu\partial_\nu\phi + \partial_\mu\partial_\alpha\phi\,\partial_\nu\partial^\alpha\phi. \tag{8.27}$$

$\mathcal{L}_2$ contains $(\partial^2\phi)^3$ interactions. We can cancel them by adding an appropriate combination of terms cubic in H, $\mathcal{L}_3 = \left(\frac{1}{2}[H][H^2] - \frac{1}{2}[H^3]\right)$.

However, the third order term we can add is not unique. At every order, there is a combination that reduces to a total derivative when expanded in ϕ's,

$$\begin{aligned}
\mathcal{L}_2^{\rm TD} &= [H^2] - [H]^2 &(8.28)\\
\mathcal{L}_3^{\rm TD} &= 3[H][H^2] - [H]^3 - 2[H^3] &(8.29)\\
\mathcal{L}_4^{\rm TD} &= [H]^4 - 6[H^2][H]^2 + 8[H^3][H] + 3[H^2]^2 - 6[H^4] &(8.30)\\
&\vdots &(8.31)
\end{aligned}$$

$\mathcal{L}_2^{\rm TD}$ is just the Fiertz-Pauli term, and the others can be thought of as higher order generalizations of it.

Thus we can add $\mathcal{L}_3^{\rm TD}$ to $\mathcal{L}_3$ with an arbitrary overall coefficient α_3. At this point, the lagrangian is $\mathcal{L} = \mathcal{L}_3 + \alpha_3\mathcal{L}_3^{\rm TD}$, which has no $(\partial^2\phi)^3$ interaction. It has $(\partial^2\phi)^4$ interactions, which can be cancelled by adding $\mathcal{L}_4 + \alpha_4\mathcal{L}_4^{\rm TD}$, where

$$\mathcal{L}_4 = \frac{1}{16}\Big[(5+24\alpha_3)[H^4]-(1+12\alpha_3)[H^2]^2-(4+24\alpha_3)[H][H^3]+12\alpha_3[H^2][H]^2\Big]. \quad (8.32)$$

This process can be repeated at all orders, and at the end there will be no terms $\sim (\partial^2\phi)^n$, and the lowest interaction scale will be due to the terms

$$\sim \frac{(\partial A)^2\partial^2\phi}{M_p m^2}, \quad \frac{(\partial A)^2(\partial^2\phi)^2}{M_p^2 m^4}, \quad \frac{(h+\phi)(\partial^2\phi)^2}{M_p m^2} \quad (8.33)$$

which are suppressed by the scale $\Lambda_3 = (M_p m^2)^{1/3}$. The cutoff has been lowered to Λ_3. The Vainstein radius will correspondingly shrink to

$$r_V^{(3)} = \left(\frac{M}{M_P}\right)^{1/3}\frac{1}{\Lambda_3} = \left(\frac{GM}{m^2}\right)^{1/3}. \quad (8.34)$$

The scale of quantum effects is now the same as the Vainstein radius,

$$r_*^{(3)} = \left(\frac{M}{M_P}\right)^{1/3}\frac{1}{\Lambda_3}. \quad (8.35)$$

It can be shown [12] that a ghost is still present around massive sources even in this Λ_3 theory. The ghost mass sinks below the cutoff at a radius parametrically larger than the Vainstein and $r_*^{(3)}$ radii,

$$r_{\rm ghost}^{(3)} \sim \left(\frac{M}{M_P}\right)^{1/2}\frac{1}{\Lambda_3} \gg r_V^{(3)} \sim r_*^{(3)} \sim \left(\frac{M}{M_P}\right)^{1/3}\frac{1}{\Lambda_3}. \quad (8.36)$$

Thus, there is a region where the ghost is lower than the cutoff but the linear classical theory is still valid. This is inconsistent unless we lower the cutoff of the effective theory so that the ghost stays above it, and we imagine that a UV completion cures the ghost.

9 Conclusions

Massive gravity stinks. Who knows whether the graviton is truly massless or not, or if predictions are continuous in the mass. If you want to make progress, try some other way to modify gravity.

References

[1] P. Van Nieuwenhuizen, "On Ghost-Free Tensor Lagrangians And Linearized Gravitation," Nucl. Phys. B **60**, 478 (1973).

[2] M. Fierz and W. Pauli, "On Relativistic Wave Equations For Particles Of Arbitrary Spin In An Electromagnetic Field," Proc. Roy. Soc. Lond. A **173**, 211 (1939).

[3] H. van Dam and M. J. Veltman, "Massive And Massless Yang-Mills And Gravitational Fields," Nucl. Phys. B **22**, 397 (1970).

[4] V. I. Zakharov, JETP Letters (Sov. Phys.) **12**, 312 (1970).

[5] A. Karch, E. Katz and L. Randall, "Absence of a VVDZ discontinuity in AdS(AdS)," JHEP **0112**, 016 (2001) [arXiv:hep-th/0106261].

[6] M. Porrati, "No van Dam-Veltman-Zakharov discontinuity in AdS space," Phys. Lett. B **498**, 92 (2001) [arXiv:hep-th/0011152].

[7] I. I. Kogan, S. Mouslopoulos and A. Papazoglou, "The m $\to$ 0 limit for massive graviton in dS(4) and AdS(4): How to circumvent the van Dam-Veltman-Zakharov discontinuity," Phys. Lett. B **503**, 173 (2001) [arXiv:hep-th/0011138].

[8] A. I. Vainshtein, "To The Problem Of Nonvanishing Gravitation Mass," Phys. Lett. B **39**, 393 (1972).

[9] M. Porrati, "Fully covariant van Dam-Veltman-Zakharov discontinuity, and absence thereof," Phys. Lett. B **534**, 209 (2002) [arXiv:hep-th/0203014].

[10] C. Deffayet, G. R. Dvali, G. Gabadadze and A. I. Vainshtein, "Nonperturbative continuity in graviton mass versus perturbative discontinuity," Phys. Rev. D **65**, 044026 (2002) [arXiv:hep-th/0106001].

[11] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, "Effective field theory for massive gravitons and gravity in theory space," Annals Phys. **305** (2003) 96 [hep-th/0210184].

[12] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, "Ghosts in massive gravity," JHEP **0509**, 003 (2005) [arXiv:hep-th/0505147].

[13] A. Gruzinov, “On the graviton mass,” astro-ph/0112246.

[14] T. Damour, I. I. Kogan and A. Papazoglou, “Spherically symmetric spacetimes in massive gravity,” Phys. Rev. D **67**, 064009 (2003) [arXiv:hep-th/0212155].

[15] D. G. Boulware and S. Deser, “Can gravitation have a finite range?,” Phys. Rev. D **6** (1972) 3368.

[16] E. C. G. Stuckelberg, Helv. Phys. Act. **30**, 209 (1957)

[17] S. Deser and R. I. Nepomechie, Annals Phys. **154**, 396 (1984).

[18] M. Henneaux, Contemp. Math. **219**, 93 (1998) [arXiv:hep-th/9712226].